

## Differential Equations

7

## 7.2

# Direction Fields and Euler's Method

# Direction Fields and Euler's Method

Unfortunately, it's impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution.

In this section we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's method).



# Direction Fields

# Direction Fields

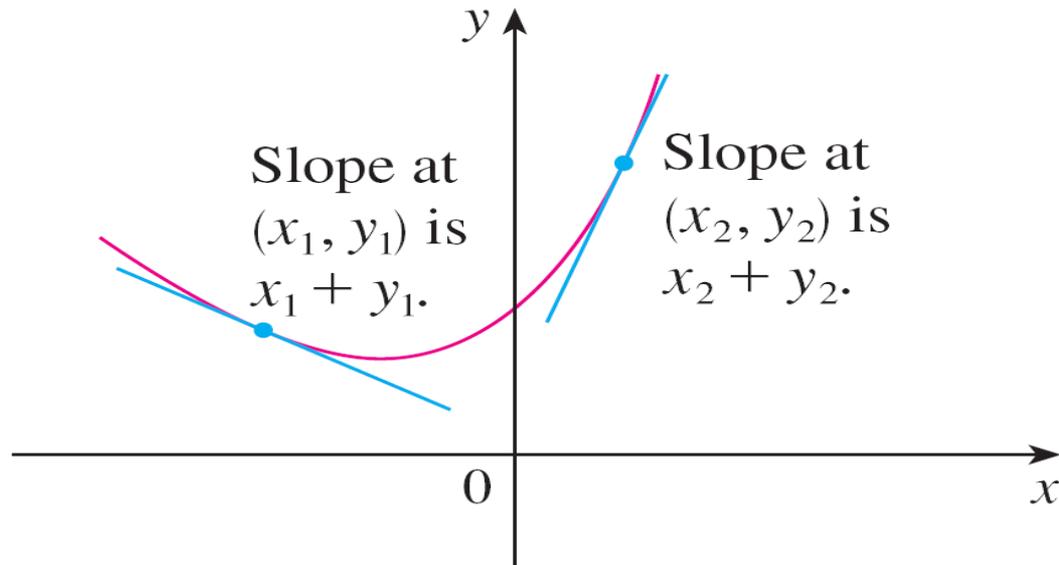
Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y \quad y(0) = 1$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means.

# Direction Fields

The equation  $y' = x + y$  tells us that the slope at any point  $(x, y)$  on the graph (called the *solution curve*) is equal to the sum of the  $x$ - and  $y$ -coordinates of the point (see Figure 1).

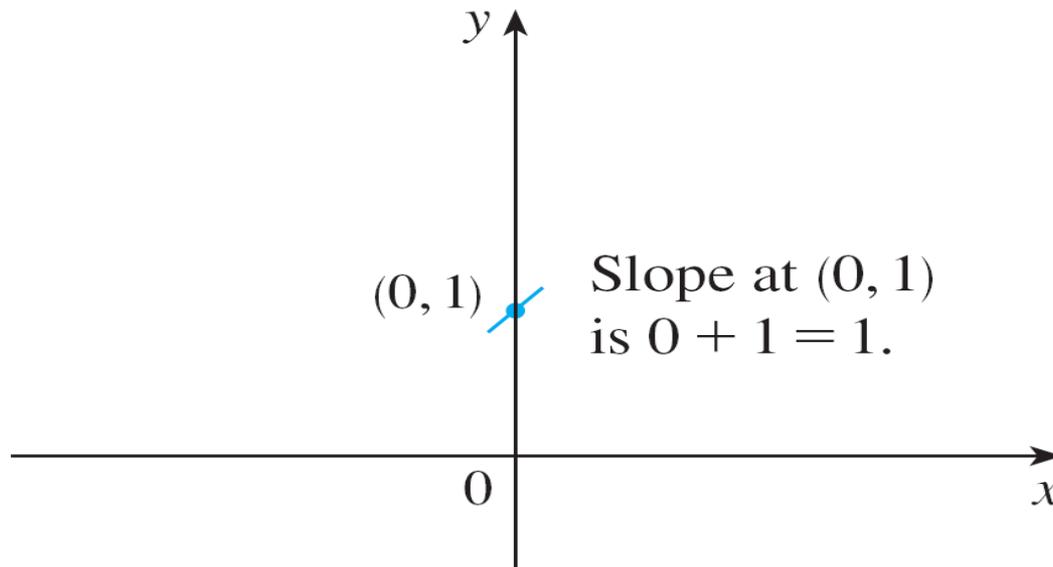


**Figure 1**

A solution of  $y' = x + y$

# Direction Fields

In particular, because the curve passes through the point  $(0, 1)$ , its slope there must be  $0 + 1 = 1$ . So a small portion of the solution curve near the point  $(0, 1)$  looks like a short line segment through  $(0, 1)$  with slope 1. (See Figure 2.)



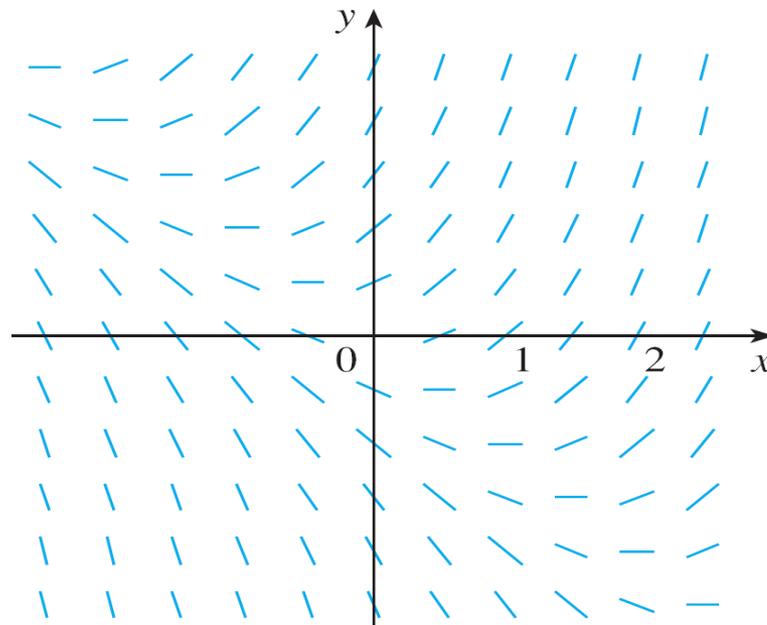
**Figure 2**

Beginning of the solution curve through  $(0, 1)$

# Direction Fields

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points  $(x, y)$  with slope  $x + y$ .

The result is called a *direction field* and is shown in Figure 3.



**Figure 3**

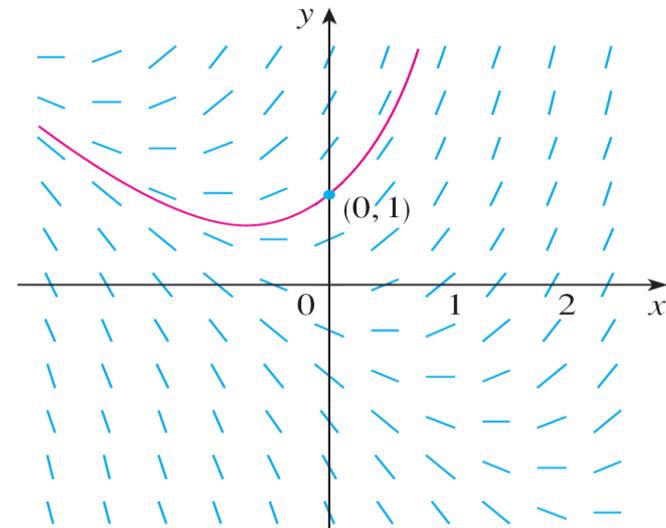
Direction field for  $y' = x + y$

# Direction Fields

For instance, the line segment at the point  $(1, 2)$  has slope  $1 + 2 = 3$ .

The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

Now we can sketch the solution curve through the point  $(0, 1)$  by following the direction field as in Figure 4.



**Figure 4**

The solution curve through  $(0, 1)$

# Direction Fields

Notice that we have drawn the curve so that it is parallel to nearby line segments.

In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

where  $F(x, y)$  is some expression in  $x$  and  $y$ .

The differential equation says that the slope of a solution curve at a point  $(x, y)$  on the curve is  $F(x, y)$ .

# Direction Fields

If we draw short line segments with slope  $F(x, y)$  at several points  $(x, y)$ , the result is called a **direction field** (or **slope field**).

These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

## Example 1 – Using a Direction Field to Sketch a Solution Curve

(a) Sketch the direction field for the differential equation

$$y' = x^2 + y^2 - 1.$$

(b) Use part (a) to sketch the solution curve that passes through the origin.

**Solution:**

(a) We start by computing the slope at several points in the following chart:

$x$	-2	-1	0	1	2	-2	-1	0	1	2	...
$y$	0	0	0	0	0	1	1	1	1	1	...
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4	...

# Example 1 – *Solution*

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Now we draw short line segments with these slopes at these points. The result is the direction field shown in Figure 5.

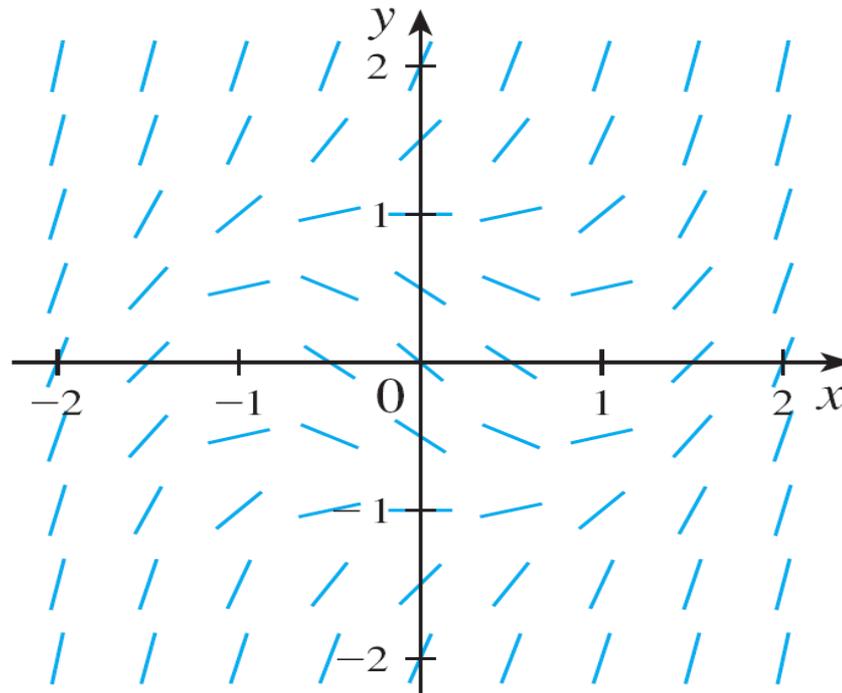


Figure 5

# Example 1 – Solution

cont'd

(b) We start at the origin and move to the right in the direction of the line segment (which has slope  $-1$ ).

We continue to draw the solution curve so that it moves parallel to the nearby line segments.

The resulting solution curve is shown in Figure 6.

Returning to the origin, we draw the solution curve to the left as well.

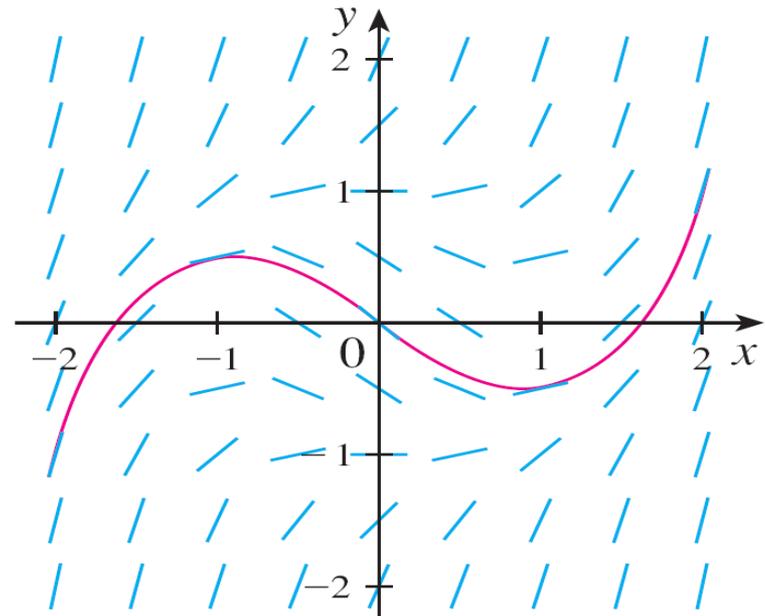


Figure 6

# Direction Fields

Now let's see how direction fields give insight into physical situations.

The simple electric circuit shown in Figure 9 contains an electromotive force (usually a battery or generator) that produces a voltage of  $E(t)$  volts (V) and a current of  $I(t)$  amperes (A) at time  $t$ .

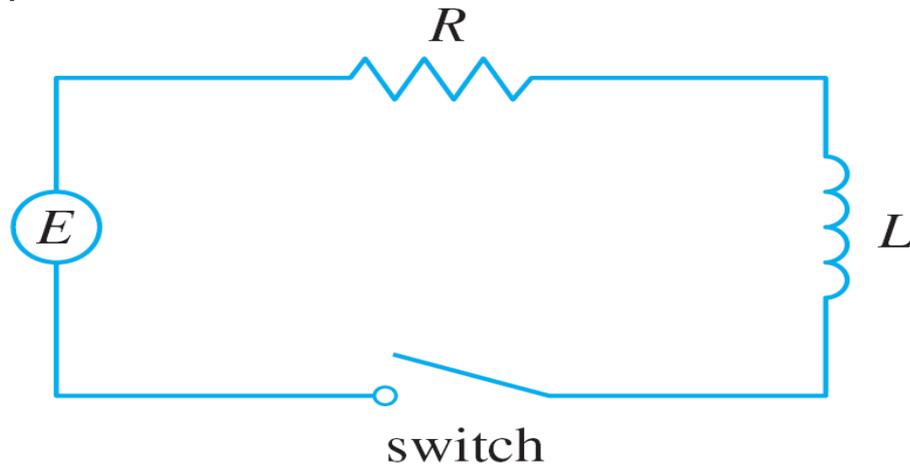


Figure 9

# Direction Fields

The circuit also contains a resistor with a resistance of  $R$  ohms ( $\Omega$ ) and an inductor with an inductance of  $L$  henries (H).

Ohm's Law gives the drop in voltage due to the resistor as  $RI$ . The voltage drop due to the inductor is  $L(di/dt)$ . One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage  $E(t)$ . Thus we have

$$\boxed{1} \quad L \frac{dI}{dt} + RI = E(t)$$

which is a first-order differential equation that models the current  $I$  at time  $t$ .

# Direction Fields

A differential equation of the form

$$y' = f(y)$$

in which the independent variable is missing from the right side, is called **autonomous**.

For such an equation, the slopes corresponding to two different points with the same  $y$ -coordinate must be equal.

This means that if we know one solution to an autonomous differential equation, then we can obtain infinitely many others just by shifting the graph of the known solution to the right or left.



# Euler's Method

# Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations.

We illustrate the method on the initial-value problem that we used to introduce direction fields:

$$y' = x + y \quad y(0) = 1$$

The differential equation tells us that  $y'(0) = 0 + 1 = 1$ , so the solution curve has slope 1 at the point  $(0, 1)$ .

As a first approximation to the solution we could use the linear approximation  $L(x) = x + 1$ .

# Euler's Method

In other words, we could use the tangent line at  $(0, 1)$  as a rough approximation to the solution curve (see Figure 12).

Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a midcourse correction by changing direction as indicated by the direction field.

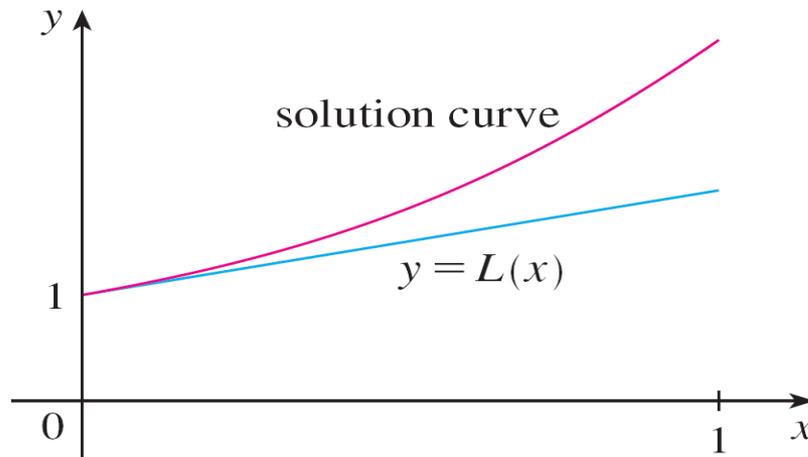


Figure 12

First Euler approximation

# Euler's Method

Figure 13 shows what happens if we start out along the tangent line but stop when  $x = 0.5$ . (This horizontal distance traveled is called the *step size*.)

Since  $L(0.5) = 1.5$ , we have  $y(0.5) \approx 1.5$  and we take  $(0.5, 1.5)$  as the starting point for a new line segment.

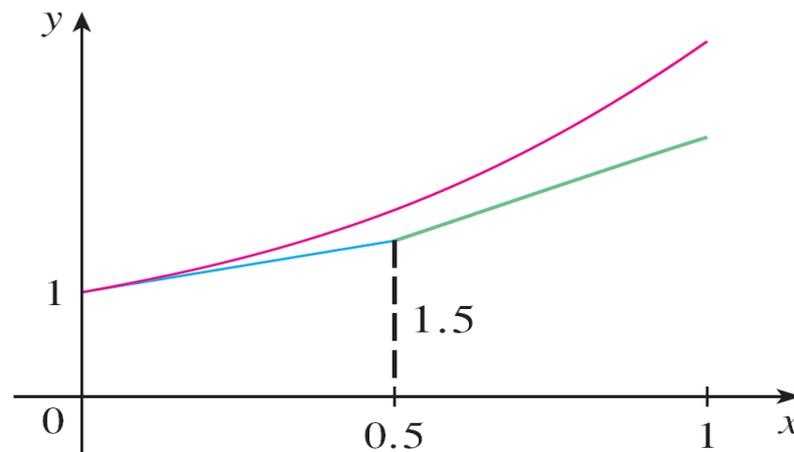


Figure 13

Euler approximation with step size 0.5

# Euler's Method

The differential equation tells us that  $y'(0.5) = 0.5 + 1.5 = 2$ , so we use the linear function

$$y = 1.5 + 2(x - 0.5) = 2x + 0.5$$

as an approximation to the solution for  $x > 0.5$  (the green segment in Figure 13).

If we decrease the step size from 0.5 to 0.25, we get the better Euler approximation shown in Figure 14.

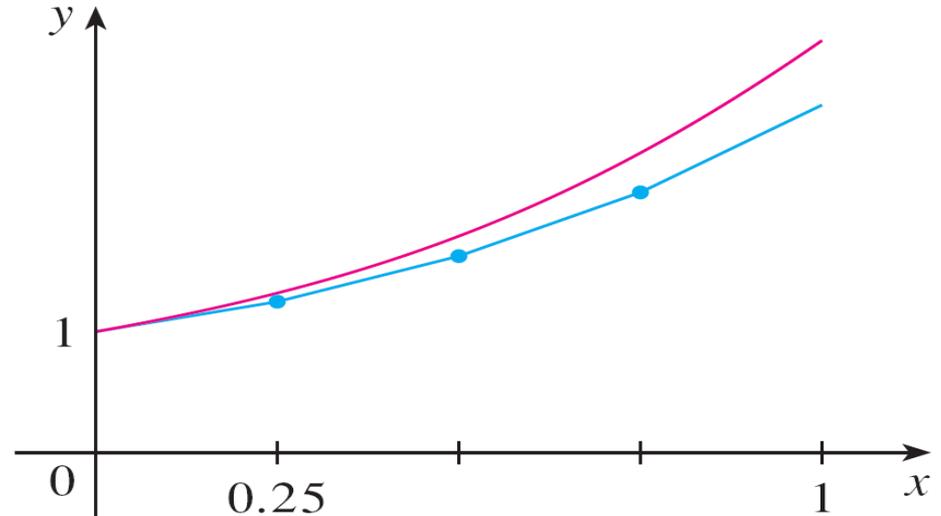


Figure 14

Euler approximation with step size 0.25

# Euler's Method

In general, Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field.

Stop after a short time, look at the slope at the new location, and proceed in that direction.

Keep stopping and changing direction according to the direction field.

Euler's method does not produce the exact solution to an initial-value problem—it gives approximations.

# Euler's Method

But by decreasing the step size (and therefore increasing the number of midcourse corrections), we obtain successively better approximations to the exact solution. (Compare Figures 12, 13, and 14.)

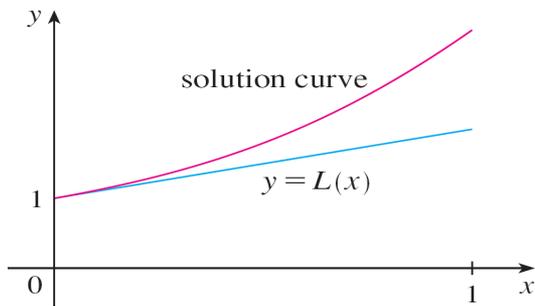


Figure 12

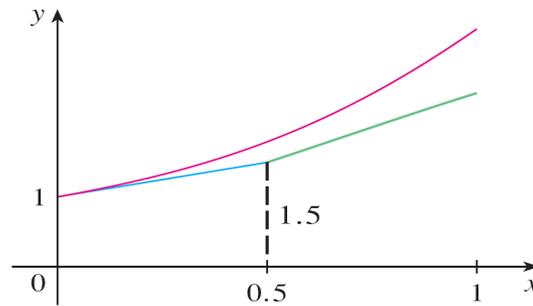


Figure 13

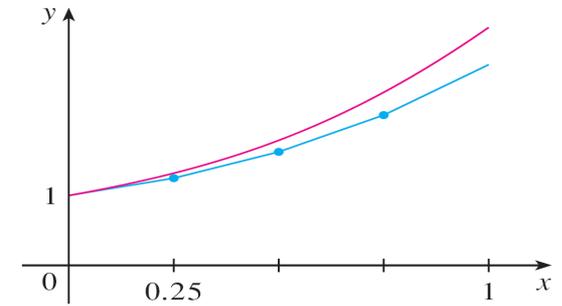


Figure 14

For the general first-order initial-value problem  $y' = F(x, y)$ ,  $y(x_0) = y_0$ , our aim is to find approximate values for the solution at equally spaced numbers  $x_0$ ,  $x_1 = x_0 + h$ ,  $x_2 = x_1 + h, \dots$ , where  $h$  is the step size.

# Euler's Method

The differential equation tells us that the slope at  $(x_0, y_0)$  is  $y' = F(x_0, y_0)$ , so Figure 15 shows that the approximate value of the solution when  $x = x_1$  is

$$y_1 = y_0 + hF(x_0, y_0)$$

Similarly,

$$y_2 = y_1 + hF(x_1, y_1)$$

In general,

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

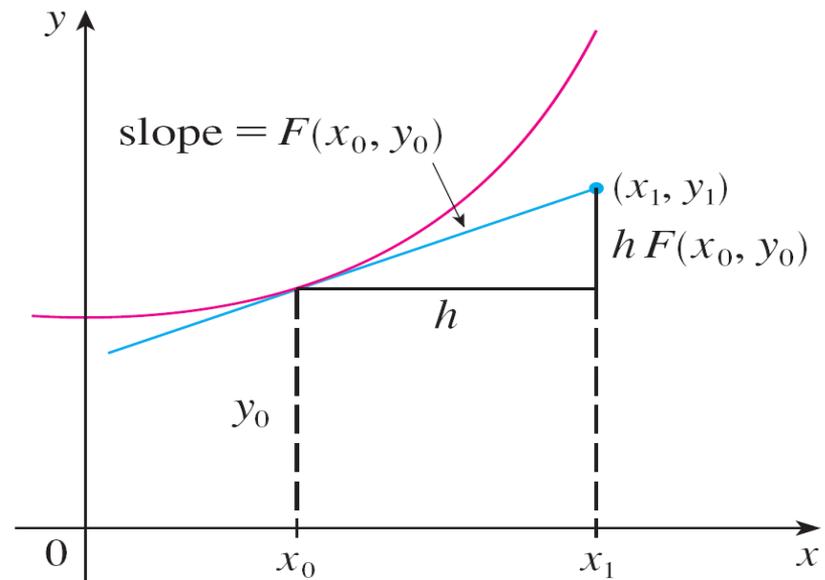


Figure 15

# Euler's Method

**Euler's Method** Approximate values for the solution of the initial-value problem  $y' = F(x, y)$ ,  $y(x_0) = y_0$ , with step size  $h$ , at  $x_n = x_{n-1} + h$ , are

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \quad n = 1, 2, 3, \dots$$

# Example 3

Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

**Solution:**

We are given that  $h = 0.1$ ,  $x_0 = 0$ ,  $y_0 = 1$ , and  $F(x, y) = x + y$ . So we have

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

This means that if  $y(x)$  is the exact solution, then  $y(0.3) \approx 1.362$ .

# Example 3 – *Solution*

cont'd

Proceeding with similar calculations, we get the values in the table:

$n$	$x_n$	$y_n$	$n$	$x_n$	$y_n$
1	0.1	1.100000	6	0.6	1.943122
2	0.2	1.220000	7	0.7	2.197434
3	0.3	1.362000	8	0.8	2.487178
4	0.4	1.528200	9	0.9	2.815895
5	0.5	1.721020	10	1.0	3.187485

# Euler's Method

For a more accurate table of values in Example 3 we could decrease the step size.

But for a large number of small steps the amount of computation is considerable and so we need to program a calculator or computer to carry out these calculations.

# Euler's Method

The following table shows the results of applying Euler's method with decreasing step size to the initial-value problem of Example 3.

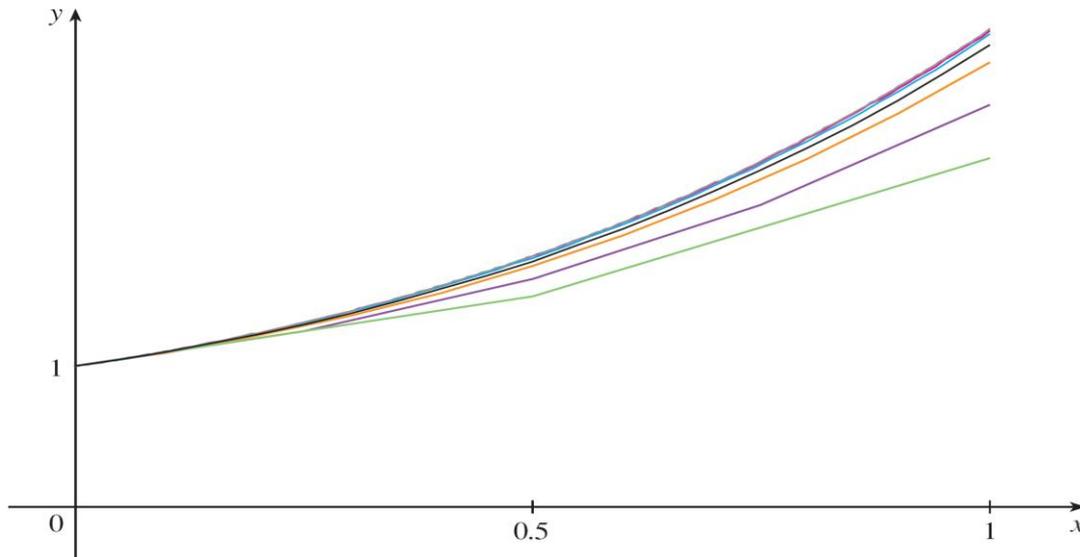
Step size	Euler estimate of $y(0.5)$	Euler estimate of $y(1)$
0.500	1.500000	2.500000
0.250	1.625000	2.882813
0.100	1.721020	3.187485
0.050	1.757789	3.306595
0.020	1.781212	3.383176
0.010	1.789264	3.409628
0.005	1.793337	3.423034
0.001	1.796619	3.433848

Notice that the Euler estimates in the table seem to be approaching limits, namely, the true values of  $y(0.5)$  and  $y(1)$ .

# Euler's Method

Figure 16 shows graphs of the Euler approximations with step sizes 0.5, 0.25, 0.1, 0.05, 0.02, 0.01, and 0.005.

They are approaching the exact solution curve as the step size  $h$  approaches 0.



**Figure 16**

Euler approximations approaching the exact solution