Differential Equations **7**

7.4 **Exponential Growth and Decay**

Exponential Growth and Decay

One of the models for population growth that we considered was based on the assumption that the population grows at a rate proportional to the size of the population:

$$\frac{dP}{dt} = kP$$

Is that a reasonable assumption? Suppose we have a population (of bacteria, for instance) with size $P = 1000$ and at a certain time it is growing at a rate of $P' = 300$ bacteria per hour.

Exponential Growth and Decay

Now let's take another 1000 bacteria of the same type and put them with the first population. Each half of the new population was growing at a rate of 300 bacteria per hour.

We would expect the total population of 2000 to increase at a rate of 600 bacteria per hour initially (provided there's enough room and nutrition).

So if we double the size, we double the growth rate. In general, it seems reasonable that the growth rate should be proportional to the size.

Exponential Growth and Decay

The same assumption applies in other situations as well. In nuclear physics, the mass of a radioactive substance decays at a rate proportional to the mass.

In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance.

In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

Exponential Growth and Decay

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

1
$$\frac{dy}{dt} = ky$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). Because it is a separable differential equation we can solve it.

Exponential Growth and Decay

$$\int \frac{dy}{y} = \int k \, dt$$

$$\ln |y| = kt + C$$

$$|y| = e^{kt+C} = e^{C}e^{kt}$$

$$y = Ae^{kt}$$

where A (= $\pm e^{C}$ or 0) is an arbitrary constant.

To see the significance of the constant A, we observe that

$$y(0) = Ae^{k \cdot 0} = A$$

Therefore A is the initial value of the function.

Exponential Growth and Decay

Because Equation 1 occurs so frequently in nature, we summarize the following.

2 The solution of the initial-value problem

$$\frac{dy}{dt} = ky \qquad y(0) = y_0$$

is

$$y(t) = y_0 e^{kt}$$

Population Growth

Population Growth

What is the significance of the proportionality constant k? In the context of population growth, we can write

3
$$\frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{1}{P}\frac{dP}{dt} = k$$

The quantity

$$\frac{1}{P}\frac{dP}{dt}$$

is the growth rate divided by the population size; it is called the **relative growth rate**.

Population Growth

According to (3), instead of saying "the growth rate is proportional to population size" we could say "the relative growth rate is constant."

Then (2) says that a population with constant relative growth rate must grow exponentially.

Notice that the relative growth rate k appears as the coefficient of t in the exponential function $y_0 e^{kt}$.

Population Growth

For instance, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is $k = 0.02$ and the population grows at a relative rate of 2% per year. If the population at time 0 is P_0, then the expression for the population is

$$P(t) = P_0 e^{0.02t}$$

Example 1 – *Modeling World Population with the Law of Natural Growth*

Assuming that the growth rate is proportional to population size, use the data in Table 1 to model the population of the world in the 20th century. What is the relative growth rate? How well does the model fit the data?

Table 1

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

Example 1 – *Solution*

We measure the time t in years and let $t = 0$ in the year 1900. We measure the population $P(t)$ in millions of people. Then the initial condition is $P(0) = 1650$. We are assuming that the growth rate is proportional to population size, so the initial-value problem is

$$\frac{dP}{dt} = kP \qquad\qquad P(0) = 1650$$

From (2) we know that the solution is

$$P(t) = 1650e^{kt}$$

Example 1 – *Solution*

cont'd

One way to estimate the relative growth rate k is to use the fact that the population in 1950 was 2560 million.

Therefore

$$P(50) = 1650e^{k(50)}$$

$$= 2560$$

We solve this equation for k:

$$e^{50k} = \frac{2560}{1650}$$

Example 1 – *Solution*

cont'd

$$k = \frac{1}{50} \ln \frac{2560}{1650}$$

$$\approx 0.0087846$$

Thus the relative growth rate is about 0.88% per year and the model becomes

$$P(t) = 1650e^{0.0087846t}$$

Example 1 – *Solution*
cont'd

Table 2 and Figure 1 allow us to compare the predictions of this model with the actual data.

Table 2

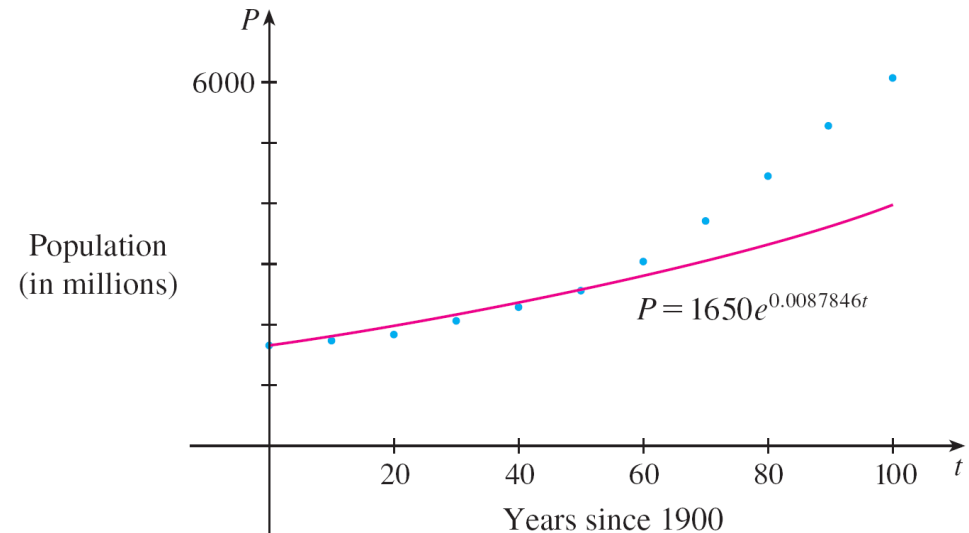| Year	Model	Population
1900	1650	1650
1910	1802	1750
1920	1967	1860
1930	2148	2070
1940	2345	2300
1950	2560	2560
1960	2795	3040
1970	3052	3710
1980	3332	4450
1990	3638	5280
2000	3972	6080

$$P = 1650e^{0.0087846t}$$

Figure 1

A possible model for world population growth

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Example 1 – *Solution* cont'd

You can see that the predictions become quite inaccurate after about 60 years.

Looking at Figure 1, we might think that we would get a better model by using the given population for 1970, instead of 1950, to estimate k. Then

$$P(70) = 1650e^{70k}$$

$$= 3710$$

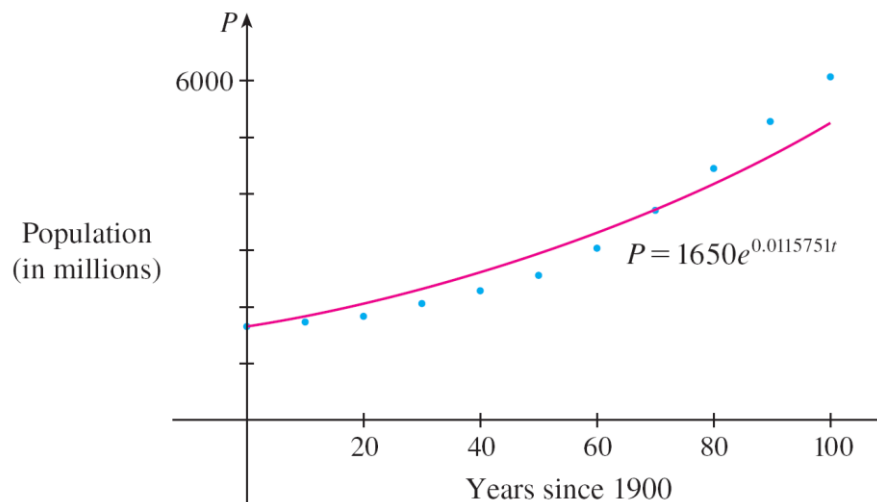$$k = \frac{1}{70} \ln \frac{3710}{1650}$$

$$\approx 0.0115751$$

Example 1 – *Solution*

The estimate for the relative growth rate is now 1.16% per year and the model is

$$P(t) = 1650e^{0.0115751t}$$

Figure 2 illustrates the second model. This exponential model is more accurate after 1970 but less accurate before 1950.

Figure 2

Another model for world population growth

Radioactive Decay

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{dm}{dt}$$

has been found experimentally to be constant. (Since dm/dt is negative, the relative decay rate is positive.) It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant.

Radioactive Decay

In other words, radioactive substances decay at a rate proportional to the remaining mass. This means that we can use (2) to show that the mass decays exponentially:

$$m(t) = m_0 e^{kt}$$

Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

Example 3

The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of $^{226}_{88}\text{Ra}$ that remains after t years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

Solution:

(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years.
Then $dm/dt = km$ and $y(0) = 100$, so (2) gives

$$m(t) = m(0)e^{kt}$$
$$= 100e^{kt}$$

Example 3 – *Solution*

In order to determine the value of *k*, we use the fact that $y(1590) = \frac{1}{2}(100)$.

Thus

$$100e^{1590k} = 50 \quad \text{so} \quad e^{1590k} = \frac{1}{2}$$

and

$$1590k = \ln \frac{1}{2}$$

$$= -\ln 2$$

$$k = -\frac{\ln 2}{1590}$$

Therefore $\quad m(t) = 100e^{-(\ln 2)t/1590}$

Example 3 – *Solution*

cont'd

We could use the fact that $e^{\ln 2} = 2$ to write the expression for $m(t)$ in the alternative form

$$m(t) = 100 \times 2^{-t/1590}$$

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590}$$
$$\approx 65 \text{ mg}$$

(c) We want to find the value of t such that $m(t) = 30$, that is,

$$100e^{-(\ln 2)t/1590} = 30 \quad \text{or} \quad e^{-(\ln 2)t/1590} = 0.3$$

Example 3 – *Solution*

We solve this equation for *t* by taking the natural logarithm of both sides:

$$-\frac{\ln 2}{1590} \, t = \ln 0.3$$

Thus

$$t = -1590 \, \frac{\ln 0.3}{\ln 2}$$

$$\approx 2762 \text{ years}$$

Newton's Law of Cooling

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.)

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant.

Newton's Law of Cooling

We could solve this equation as a separable differential equation, but an easier method is to make the change of variable $y(t) = T(t) - T_s$.

Because T_s is constant, we have $y'(t) = T'(t)$ and so the equation becomes

$$\frac{dy}{dt} = ky$$

We can then use (2) to find an expression for y, from which we can find T.

Example 4 – *Using Newton's Law of Cooling to Predict Temperatures*

A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.

(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

Solution:

(a) Let $T(t)$ be the temperature of the soda after t minutes. The surrounding temperature is $T_s = 44°F$, so Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - 44)$$

Example 4 – *Solution*

cont'd

If we let $y = T - 44$, then $y(0) = T(0) - 44 = 72 - 44 = 28$, so y is a solution of the initial-value problem

$$\frac{dy}{dt} = ky \qquad y(0) = 28$$

and by (2) we have

$$y(t) = y(0)e^{kt}$$
$$= 28e^{kt}$$

We are given that $T(30) = 61$, so $y(30) = 61 - 44 = 17$ and

$$28e^{30k} = 17$$

$$e^{30k} = \frac{17}{28}$$

Example 4 – *Solution*

cont'd

Taking logarithms, we have

$$k = \frac{\ln\left(\frac{17}{28}\right)}{30}$$

$$\approx -0.01663$$

Thus

$$y(t) = 28e^{-0.01663t}$$

$$T(t) = 44 + 28e^{-0.01663t}$$

$$T(60) = 44 + 28e^{-0.01663(60)}$$

$$\approx 54.3$$

So after another half hour the pop has cooled to about 54°F.

Example 4 – *Solution*

cont'd

(b) We have $T(t) = 50$ when

$$44 + 28e^{-0.01663t} = 50$$

$$e^{-0.01663t} = \frac{6}{28}$$

$$t = \frac{\ln\left(\frac{6}{28}\right)}{-0.01663}$$

$$\approx 92.6$$

The pop cools to 50°F after about 1 hour 33 minutes.

Continuously Compounded Interest

Example 5

If $1000 is invested at 6% interest, compounded annually, then after 1 year the investment is worth $1000(1.06) = $1060, after 2 years it's worth $[1000(1.06)]1.06 = $1123.60, and after t years it's worth $1000(1.06)^t.

In general, if an amount A_0 is invested at an interest rate r (r = 0.06 in this example), then after t years it's worth $A_0(1 + r)^t$. Usually, however, interest is compounded more frequently, say, n times a year.

Example 5

cont'd

Then in each compounding period the interest rate is *r/n* and there are *nt* compounding periods in *t* years, so the value of the investment is

$$A_0\left(1 + \frac{r}{n}\right)^{nt}$$

For instance, after 3 years at 6% interest a $1000 investment will be worth

$1000(1.06)^3 = 1191.02 with annual compounding

$1000(1.03)^6 = 1194.05 with semiannual compounding

$1000(1.015)^{12} = 1195.62 with quarterly compounding

Example 5

cont'd

$1000(1.005)^{36} = 1196.68 with monthly compounding

$$\$1000\left(1 + \frac{0.06}{365}\right)^{365 \cdot 3} = \$1197.20 \quad \text{with daily compounding}$$

You can see that the interest paid increases as the number of compounding periods (n) increases.

If we let $n \rightarrow \infty$, then we will be compounding the interest **continuously** and the value of the investment will be

$$A(t) = \lim_{n \to \infty} A_0\left(1 + \frac{r}{n}\right)^{nt}$$

Example 5

cont'd

$$= \lim_{n \to \infty} A_0 \left[\left(1 + \frac{r}{n} \right)^{n/r} \right]^{rt}$$

$$= A_0 \left[\lim_{n \to \infty} \left(1 + \frac{r}{n} \right)^{n/r} \right]^{rt}$$

$$= A_0 \left[\lim_{m \to \infty} \left(1 + \frac{1}{m} \right)^{m} \right]^{rt} \qquad \text{(where } m = n/r \text{)}$$

But the limit in this expression is equal to the number *e*.

Example 5

cont'd

So with continuous compounding of interest at interest rate r, the amount after t years is

$$A(t) = A_0 e^{rt}$$

If we differentiate this equation, we get

$$\frac{dA}{dt} = rA_0 e^{rt} = rA(t)$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Example 5

cont'd

Returning to the example of $1000 invested for 3 years at 6% interest, we see that with continuous compounding of interest the value of the investment will be

$$A(3) = \$1000e^{(0.06)3}$$

$$= \$1000e^{0.18}$$

$$= \$1197.22$$

Notice how close this is to the amount we calculated for daily compounding, $1197.20. But the amount is easier to compute if we use continuous compounding.