Infinite Sequences and Series

8

8.1 Sequences

Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the *first term,* a_2 is the *second term,* and in general a_n is the *nth term.* We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers.

Sequences

But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Example 1 – *Describing Sequences*

Some sequences can be defined by giving a formula for the *n*th term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that *n* doesn't have to start at 1.

(a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_n = \frac{n}{n+1}$ $\quad \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$

(b) $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$ $\quad a_n = \frac{(-1)^n(n+1)}{3^n}$

$$\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$$

Example 1

cont'd

(c) $\left\{\sqrt{n-3}\,\right\}_{n=3}^{\infty}$ $\qquad a_n = \sqrt{n-3},\ n \geqslant 3$

$$\left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$$

(d) $\left\{\cos \frac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\qquad a_n = \cos \frac{n\pi}{6},\ n \geqslant 0$

$$\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos \frac{n\pi}{6}, \ldots\right\}$$

Sequences

A sequence such as the one in Example 1(a), $a_n = n/(n + 1)$, can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2.

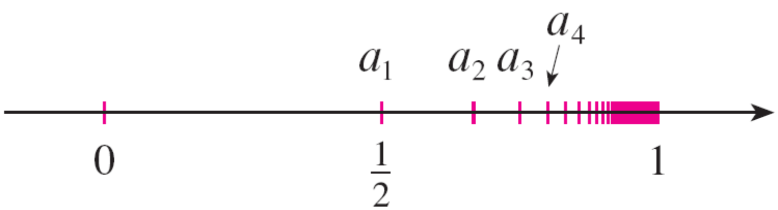

Figure 1

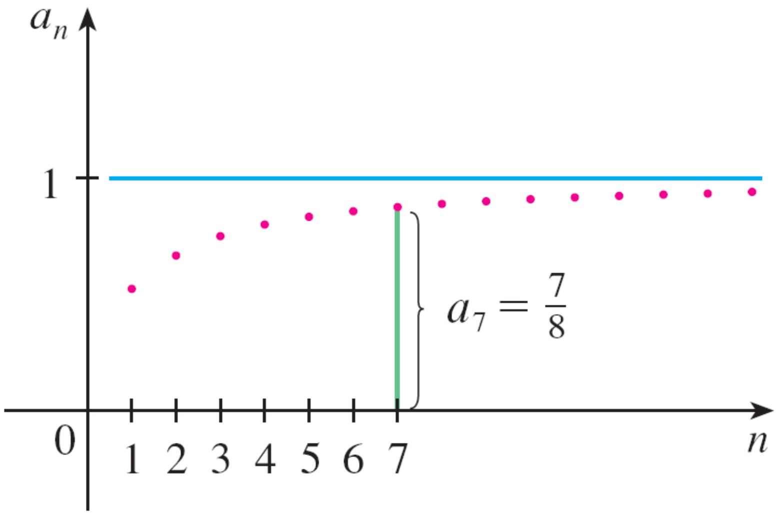

Figure 2

Sequences

Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \quad (2, a_2) \quad (3, a_3) \quad \ldots \quad (n, a_n) \quad \ldots$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n + 1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large.

Sequences

We indicate this by writing

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n \to \infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large.

Sequences

Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity.

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Sequences

Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit *L*.

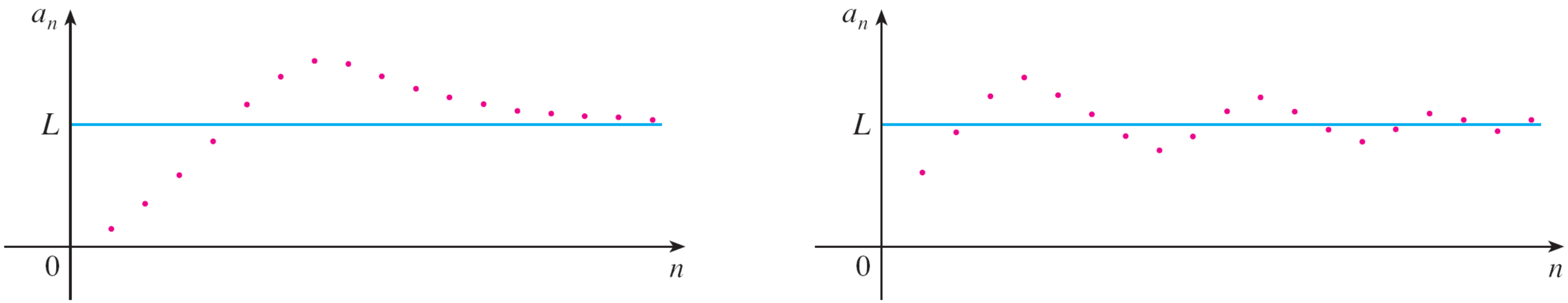

Figure 3

Graphs of two sequences with $\lim_{n \to \infty} a_n = L$

You will see that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that *n* is required to be an integer.

Sequences

Thus we have the following theorem, which is illustrated by Figure 4.

2 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

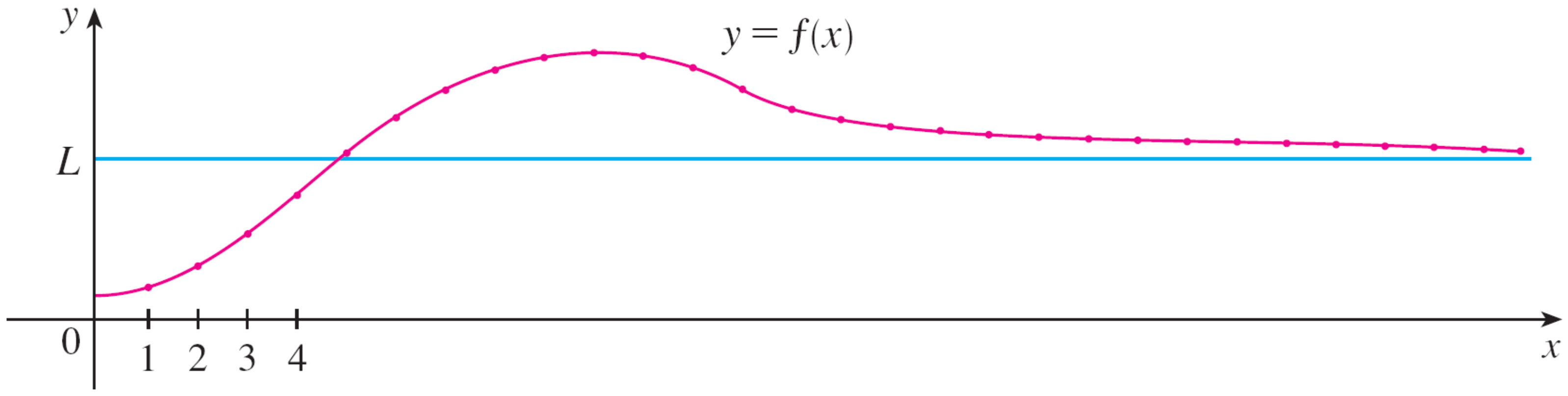

Figure 4

Sequences

In particular, since we know that $\lim_{x \to \infty}(1/x^r) = 0$ when $r > 0$, we have

3 $$\lim_{n \to \infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation

$$\lim_{n \to \infty} a_n = \infty$$

In this case the sequence $\{a_n\}$ is divergent, but in a special way. We say that $\{a_n\}$ diverges to ∞.

The Limit Laws also hold for the limits of sequences and their proofs are similar.

Sequences

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Sequences

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 5).

Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$, then $\lim_{n \to \infty} b_n = L$.

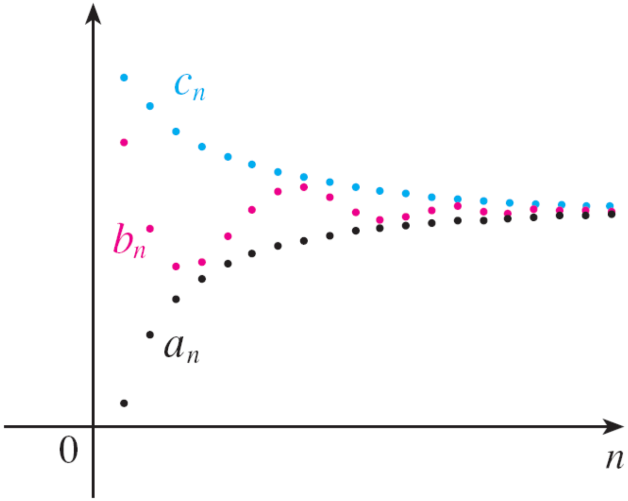

Figure 5

The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

Sequences

Another useful fact about limits of sequences is given by the following theorem, which follows from the Squeeze Theorem because $-|a_n| \le a_n \le |a_n|$.

4 Theorem If $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$.

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent.

5 Theorem If $\lim_{n \to \infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n \to \infty} f(a_n) = f(L)$$

Example 10 – *Limit of a Geometric Sequence*

For what values of r is the sequence $\{r^n\}$ convergent?

Solution:

We know that $\lim_{x\to\infty} a^x = \infty$ for $a > 1$ and $\lim_{x\to\infty} a^x = 0$ for $0 < a < 1$. Therefore, putting $a = r$ and using Theorem 2, we have

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

For the cases $r = 1$ and $r = 0$ we have

$$\lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1 \quad \text{and} \quad \lim_{n\to\infty} 0^n = \lim_{n\to\infty} 0 = 0$$

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

and therefore $\lim_{n\to\infty} r^n = 0$ by Theorem 4.

Example 10 – *Solution*

cont'd

If $r \leq -1$, then $\{r^n\}$ diverges. Figure 9 shows the graphs for various values of r. (The case $r = -1$ is shown in Figure 6.)

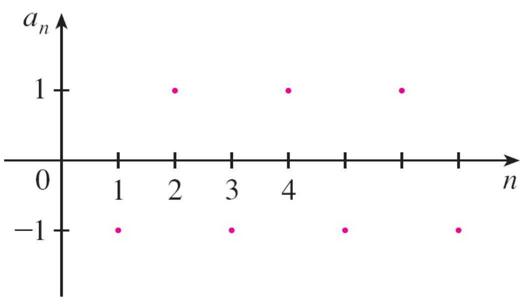

Figure 6

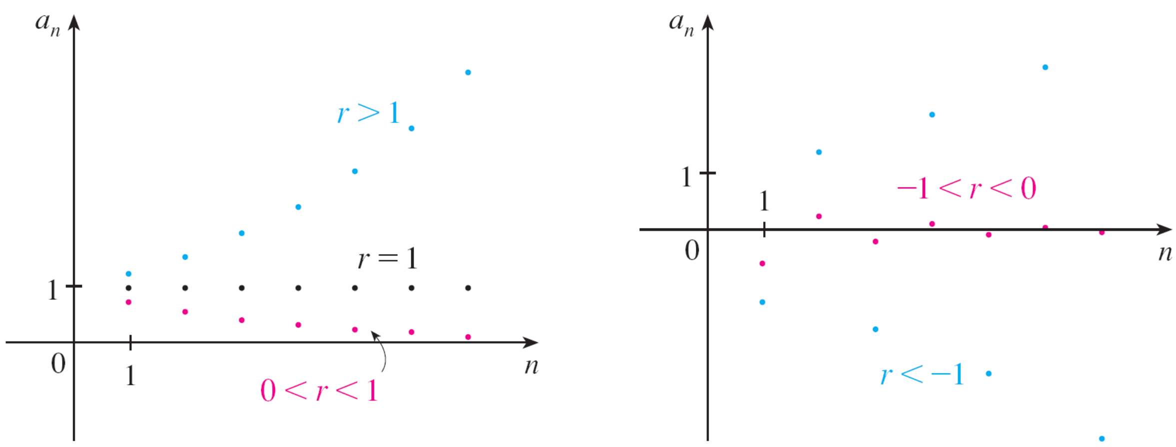

Figure 9

The sequence $a_n = r^n$

Sequences

The results of Example 10 are summarized as follows.

7 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Sequences

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = n/(n + 1)$ is bounded because $0 < a_n < 1$ for all n.

Sequences

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leq a_n \leq 1$ but is divergent,] and not every monotonic sequence is convergent ($a_n = n \to \infty$).

But if a sequence is both bounded *and* monotonic, then it must be convergent.

Sequences

This fact is stated without proof as Theorem 8, but intuitively you can understand why it is true by looking at Figure 10.

8 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

If $\{a_n\}$ is increasing and $a_n \leq M$ for all n, then the terms are forced to crowd together and approach some number L.

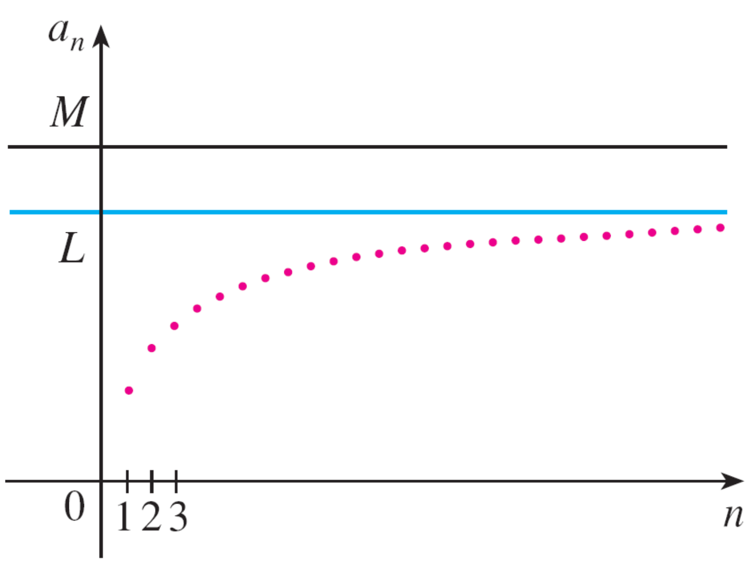

Figure 10