Infinite Sequences and Series

8

8.4 Other Convergence Tests

Alternating Series

Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

We see from these examples that the *n*th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

Alternating Series

The following test says that if the terms of an alternating series decrease to 0 in absolute value, then the series converges.

The Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

Example 1 – *Using the Alternating Series Test*

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(i) $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$

(ii) $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$

so the series is convergent by the Alternating Series Test.

Alternating Series

The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$.

The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term.

Alternating Series Estimation Theorem If $s = \Sigma\,(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

$$\text{(i)}\ \ b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii)}\ \ \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leqslant b_{n+1}$$

Absolute Convergence

Absolute Convergence

Given any series $\Sigma\, a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

Definition A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

Notice that if $\Sigma\, a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence.

Example 5 – *Determining Absolute Convergence*

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent *p*-series ($p = 2$).

Example 6 – *A Series that is Convergent but not Absolutely Convergent*

We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent (see Example 1), but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (p-series with $p = 1$) and is therefore divergent.

Absolute Convergence

Example 6 shows that it is possible for a series to be convergent but not absolutely convergent.

However, Theorem 1 shows that absolute convergence implies convergence.

1 **Theorem** If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

To see why Theorem 1 is true, observe that the inequality

$$0 \le a_n + |a_n| \le 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$.

Absolute Convergence

If $\Sigma\, a_n$ is absolutely convergent, then $\Sigma\, |a_n|$ is convergent, so $\Sigma\, 2|a_n|$ is convergent.

Therefore, by the Comparison Test, $\Sigma\, (a_n + |a_n|)$ is convergent. Then

$$\Sigma\, a_n = \Sigma\, (a_n + |a_n|) - \Sigma\, |a_n|$$

is the difference of two convergent series and is therefore convergent.

The Ratio Test

The Ratio Test

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

Example 8 – *Using the Ratio Test*

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

Solution:

We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}} \right|$$

$$= \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

Example 8 – *Solution*

cont'd

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.