Infinite Sequences and Series

8

8.5 Power Series

Power Series

A **power series** is a series of the form

$$\boxed{1} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence.

A power series may converge for some values of x and diverge for other values of x.

Power Series

The sum of the series is a function

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x_n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Power Series

More generally, a series of the form

$$\boxed{2} \qquad \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series in** $(x - a)$ or a **power series centered at *a*** or a **power series about *a***.

Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 2 we have adopted the convention that $(x - a)^0 = 1$ even when $x = a$.

Notice also that when $x = a$ all of the terms are 0 for $n \geq 1$ and so the power series (2) always converges when $x = a$.

Example 1 – *A Power Series that Converges Only at its Center*

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution:

We use the Ratio Test. If we let a_n, as usual, denote the nth term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

$$= \lim_{n\to\infty}(n+1)|x|$$

$$= \infty$$

By the Ratio Test, the series diverges when $x \neq 0$.

Thus the given series converges only when $x = 0$.

Power Series

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry.

In particular, the sum of the power series, $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, is called a **Bessel function.**

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function as the sum of a series we mean that, for every real number *x*,

$$J_0(x) = \lim_{n \to \infty} s_n(x) \qquad \text{where} \qquad s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$$

Power Series

The first few partial sums are

$$s_0(x) = 1 \qquad s_1(x) = 1 - \frac{x^2}{4} \qquad s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} \qquad s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$

Figure 1 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function J_0, but notice that the approximations become better when more terms are included.

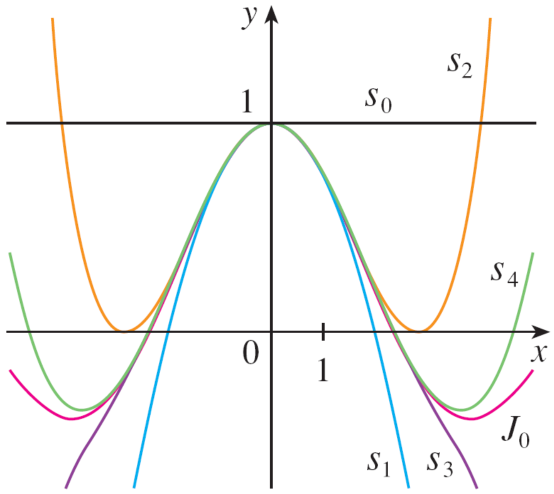

Figure 1

Partial sums of the Bessel function J_0

Power Series

Figure 2 shows a more complete graph of the Bessel function.

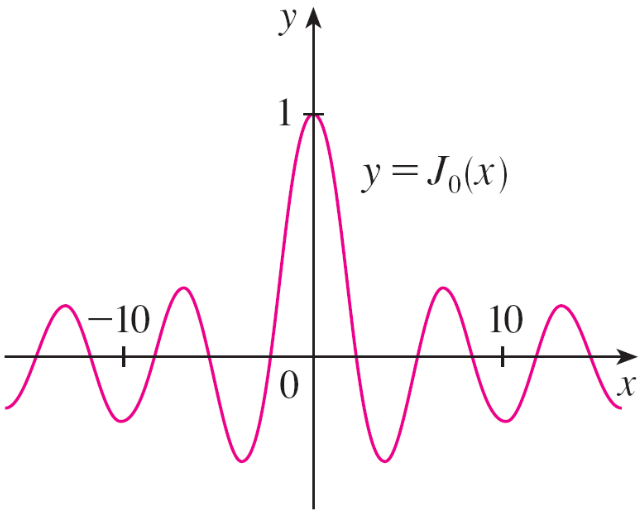

Figure 2

For the power series that we have looked at so far, the set of values of *x* for which the series is convergent has always turned out to be an interval.

Power Series

The following theorem says that this is true in general.

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number *R* in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

The **interval of convergence** of a power series is the interval that consists of all values of *x* for which the series converges.

Power Series

In case (i) the interval consists of just a single point *a*.

In case (ii) the interval is $(-\infty, \infty)$.

In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$.

When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints.

Power Series

Thus in case (iii) there are four possibilities for the interval of convergence:

$(a - R, a + R)$ $(a - R, a + R]$ $[a - R, a + R)$ $[a - R, a + R]$

The situation is illustrated in Figure 3.

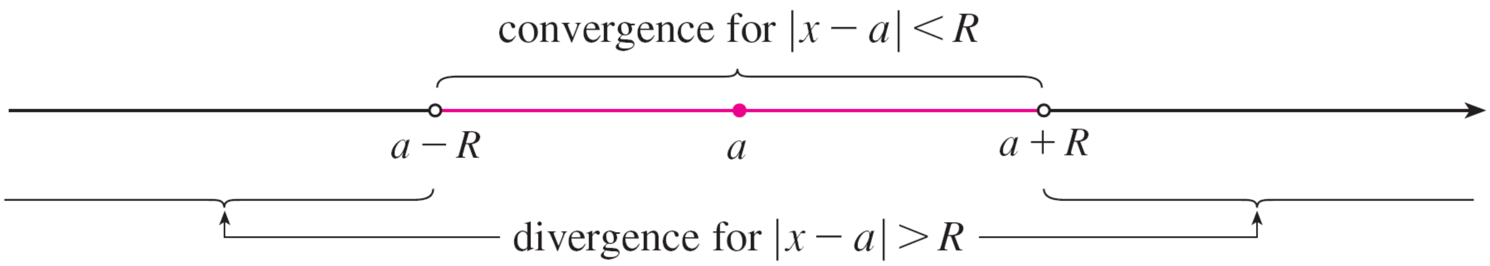

Figure 3

Power Series

We summarize here the radius and interval of convergence for each of the examples already considered in this section.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$