Vectors and the Geometry of Space

9

9.2 Vectors

Vectors

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction.

A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector.

We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter $(\vec{v})$.

Vectors

For instance, suppose a particle moves along a line segment from point *A* to point *B*.

The corresponding **displacement vector v**, shown in Figure 1, has **initial point** *A* (the tail) and **terminal point** *B* (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$

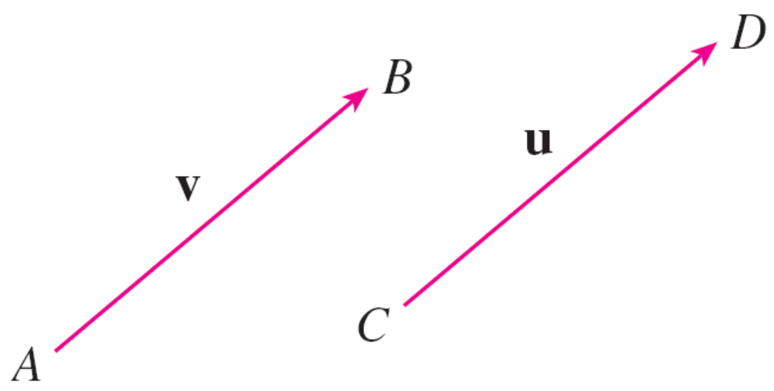


Figure 1

Equivalent vectors

Vectors

Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as **v** even though it is in a different position.

We say that **u** and **v** are **equivalent** (or **equal**) and we write **u** = **v**.

The **zero vector**, denoted by **0**, has length 0. It is the only vector with no specific direction.

Combining Vectors

Combining Vectors

Suppose a particle moves from *A* to *B*, so its displacement vector is $\overrightarrow{AB}$. Then the particle changes direction and moves from *B* to *C*, with displacement vector $\overrightarrow{BC}$ as in Figure 2.

The combined effect of these displacements is that the particle has moved from *A* to *C*.

The resulting displacement vector $\overrightarrow{AC}$ is called the *sum* of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

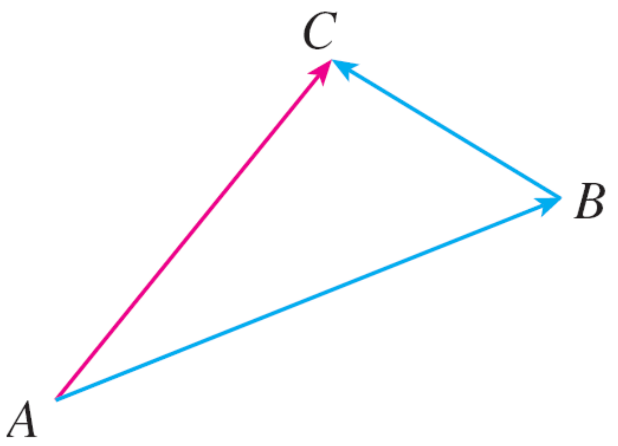


Figure 2

Combining Vectors

In general, if we start with vectors **u** and **v**, we first move **v** so that its tail coincides with the tip of **u** and define the sum of **u** and **v** as follows.

Definition of Vector Addition If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.

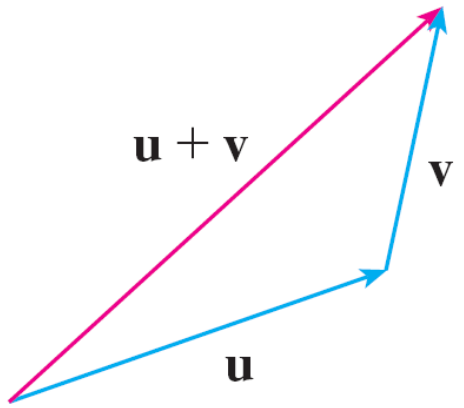


Figure 3

The Triangle Law

Combining Vectors

In Figure 4 we start with the same vectors **u** and **v** as in Figure 3 and draw another copy of **v** with the same initial point as **u**.

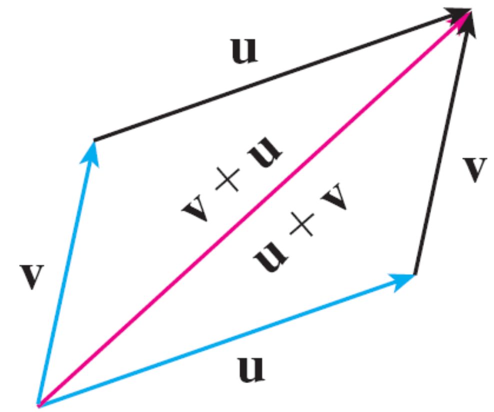


Figure 4

The Parallelogram Law

Completing the parallelogram, we see that **u** + **v** = **v** + **u**.

This also gives another way to construct the sum: If we place **u** and **v** so they start at the same point, then **u** + **v** lies along the diagonal of the parallelogram with **u** and **v** as sides. (This is called the **Parallelogram Law**.)

Example 1

Draw the sum of the vectors **a** and **b** shown in Figure 5.

Figure 5

Solution:

First we translate **b** and place its tail at the tip of **a**, being careful to draw a copy of **b** that has the same length and direction.

Example 1 – *Solution*

cont'd

Then we draw the vector **a** + **b** [see Figure 6(a)] starting at the initial point of **a** and ending at the terminal point of the copy of **b**.

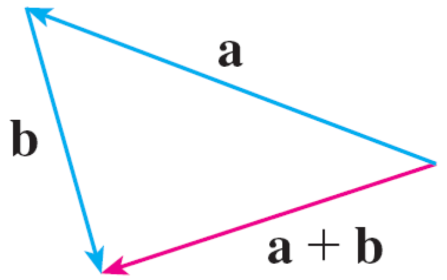


Figure 6(a)

Alternatively, we could place **b** so it starts where **a** starts and construct **a** + **b** by the Parallelogram Law as in Figure 6(b).

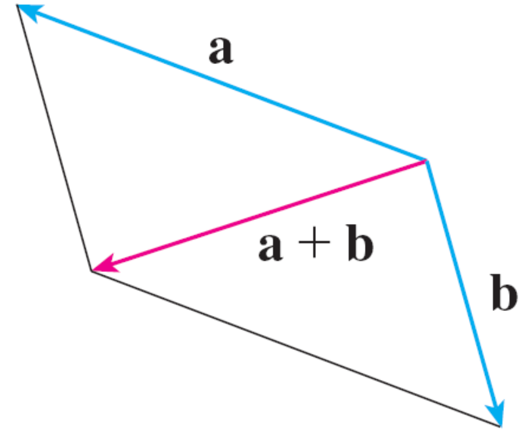


Figure 6(b)

Combining Vectors

It is possible to multiply a vector by a real number *c*. (In this context we call the real number *c* a **scalar** to distinguish it from a vector.)

For instance, we want 2**v** to be the same vector as **v** + **v**, which has the same direction as **v** but is twice as long. In general, we multiply a vector by a scalar as follows.

> **Definition of Scalar Multiplication** If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Combining Vectors

This definition is illustrated in Figure 7.

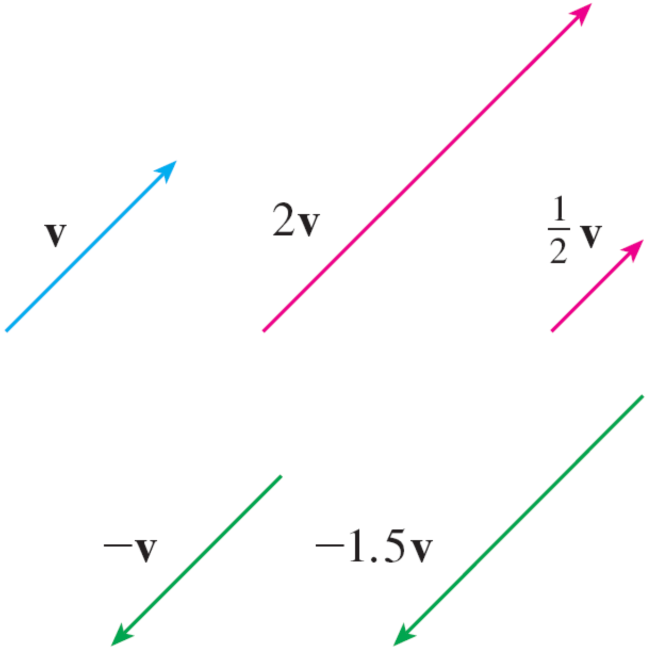


Figure 7

Scalar multiples of **v**

We see that real numbers work like scaling factors here; that’s why we call them scalars.

Combining Vectors

Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another.

In particular, the vector **–v** = (–1)**v** has the same length as **v** but points in the opposite direction. We call it the **negative** of **v**.

By the **difference u** – **v** of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Combining Vectors

So we can construct **u** – **v** by first drawing the negative of **v**, –**v**, and then adding it to **u** by the Parallelogram Law as in Figure 8(a).

Alternatively, since **v** + (**u** – **v**) = **u**, the vector **u** – **v**, when added to **v**, gives **u**. So we could construct **u** – **v** as in Figure 8(b) by means of the Triangle Law.

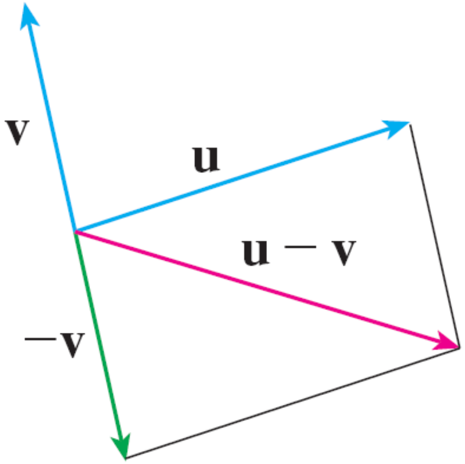


Figure 8(a)

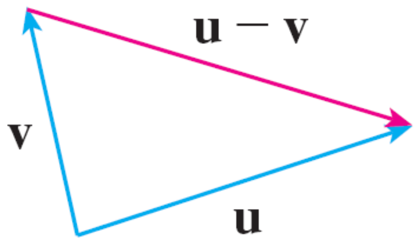


Figure 8(b)

Drawing **u – v**

Components

Components

For some purposes it's best to introduce a coordinate system and treat vectors algebraically.

If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (see Figure 11).

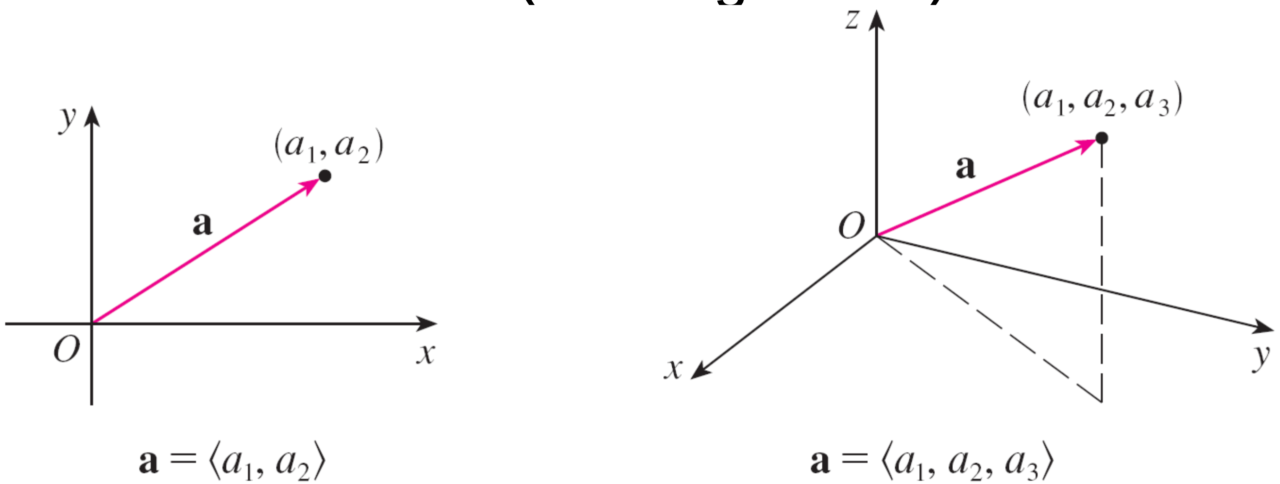


Figure 11

Components

These coordinates are called the **components** of **a** and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

For instance, the vectors shown in Figure 12 are all equivalent to the vector $\overrightarrow{OP} = \langle 3, 2 \rangle$ whose terminal point is $P(3, 2)$.

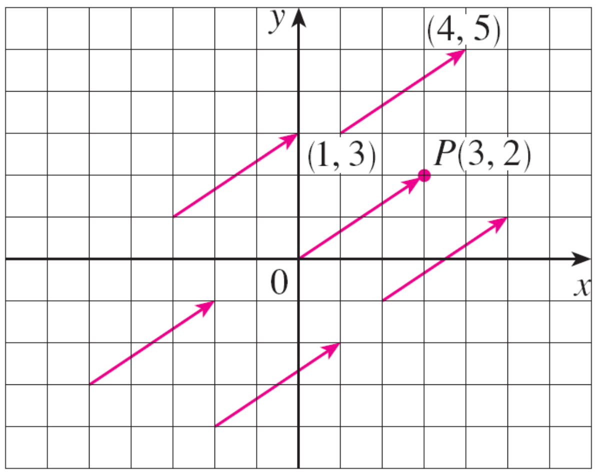


Figure 12

Representations of the vector **a** = ⟨3, 2⟩

Components

What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward.

We can think of all these geometric vectors as **representations** of the algebraic vector **a** = $\langle 3, 2 \rangle$.

The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

Components

In three dimensions, the vector **a** = $\overrightarrow{OP}$ = $\langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. (See Figure 13.)

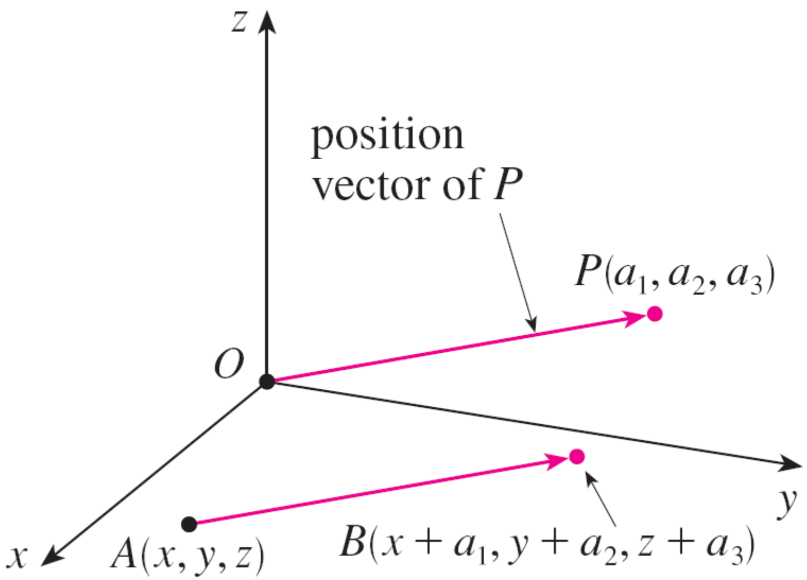


Figure 13

Representations of **a** = $\langle a_1, a_2, a_3 \rangle$

Components

Let's consider any other representation $\overrightarrow{AB}$ of **a**, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$.

Then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$.

Thus we have the following result.

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 3 – *Representing the Displacement Vector from One Point to Another*

Find the vector represented by the directed line segment with initial point *A*(2, –3, 4) and terminal point *B*(–2, 1, 1).

Solution:

By (1), the vector corresponding to $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle$$

$$= \langle -4, 4, -3 \rangle$$

Components

The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by the symbol |**v**| or ||**v**||. By using the distance formula to compute the length of a segment *OP*, we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Components

How do we add vectors algebraically? Figure 14 shows that if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive.

In other words, *to add algebraic vectors we add their components*. Similarly, *to subtract vectors we subtract components*.

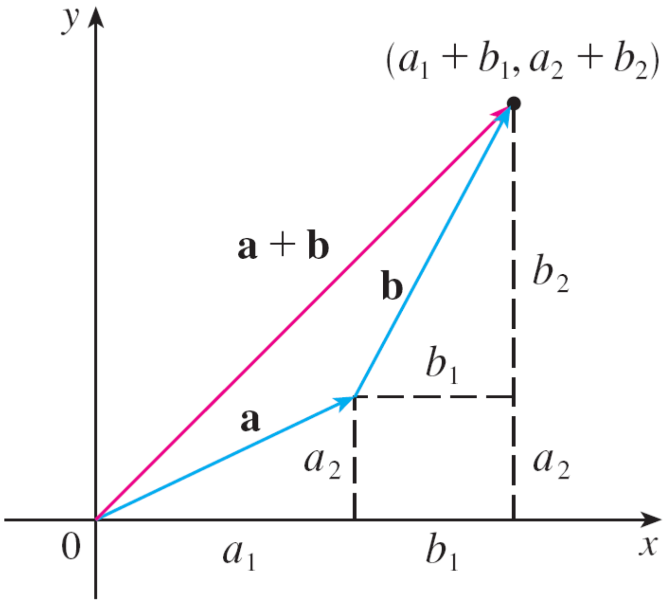


Figure 14

Components

From the similar triangles in Figure 15 we see that the components of $c\mathbf{a}$ are ca_1 and ca_2.

So *to multiply a vector by a scalar we multiply each component by that scalar*.

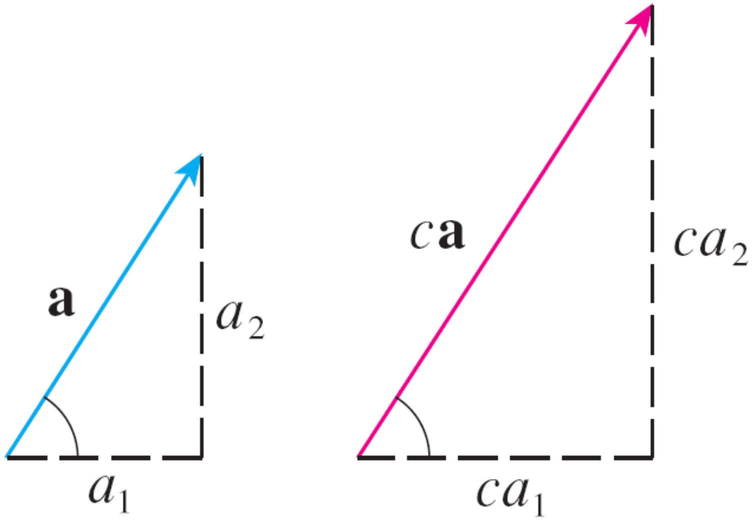


Figure 15

Components

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Components

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors.

More generally, we will consider the set V_n of all n-dimensional vectors.

An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of **a**.

Components

Addition and scalar multiplication are defined in terms of components just as for the cases n = 2 and n = 3.

Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Components

Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

Then **i**, **j**, and **k** are vectors that have length 1 and point in the directions of the positive *x*-, *y*-, and *z*-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (See Figure 17.)

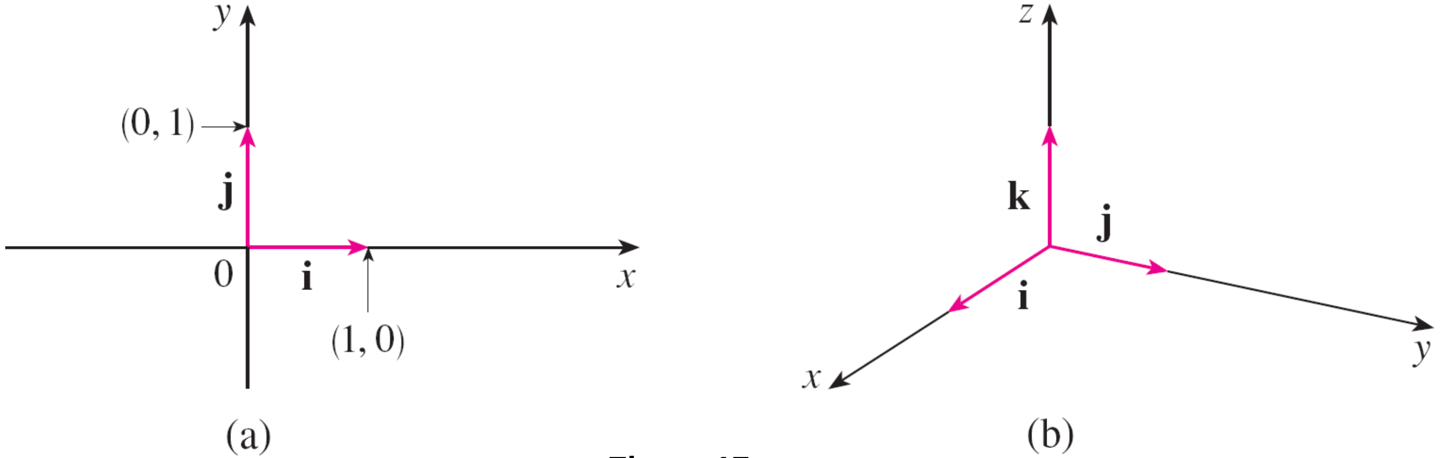


Figure 17

Standard basis vectors in V_2 and V_3

Components

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$
$$= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$$

2 $\quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Thus any vector in V_3 can be expressed in terms of the **standard basis vectors i**, **j**, and **k**. For instance,

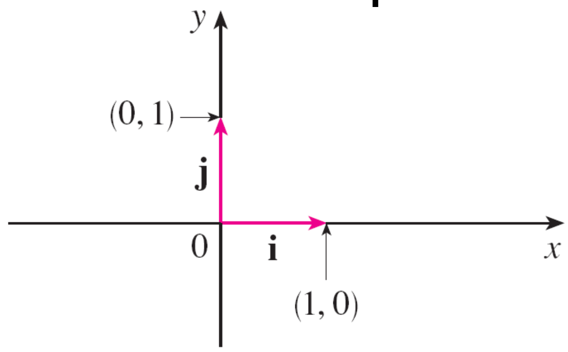
$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

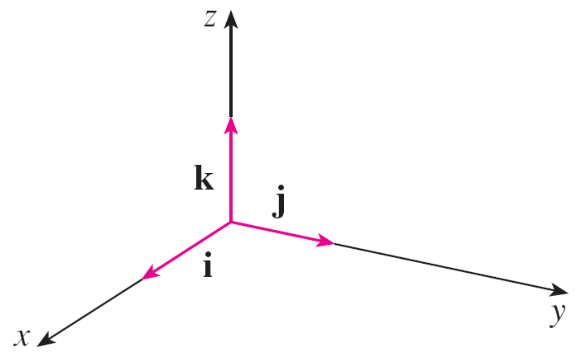
3 $\quad \mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$

Components

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.



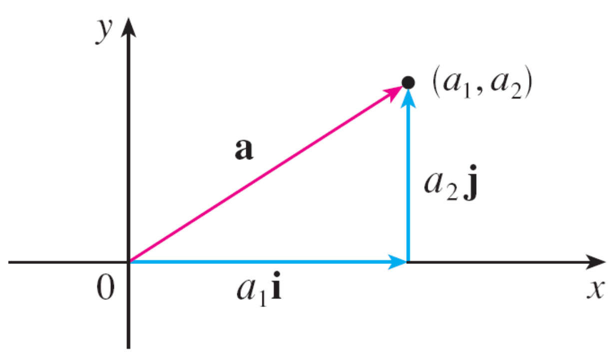
(a)



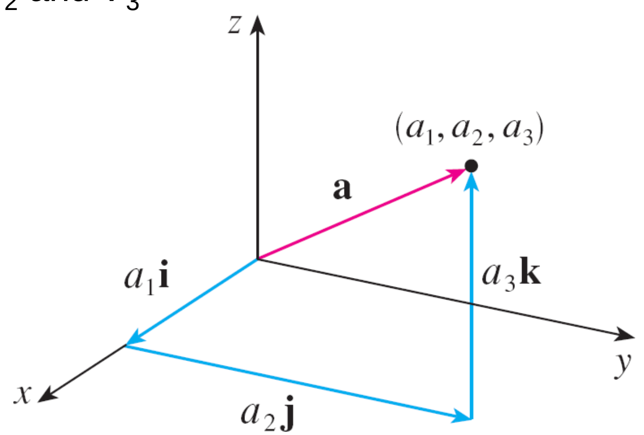
(b)

Figure 17

Standard basis vectors in V_2 and V_3



(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$



(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Figure 18

Components

A **unit vector** is a vector whose length is 1. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

4
$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

In order to verify this, we let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so **u** has the same direction as **a**. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

Applications

Applications

Vectors are useful in many aspects of physics and engineering. Here we look at forces.

A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction.

If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

Example 7

A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.

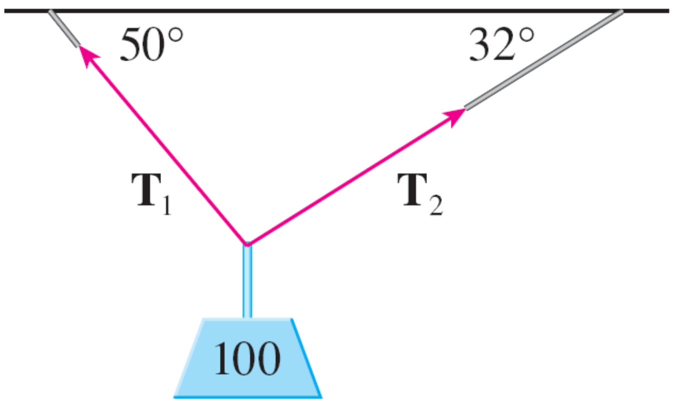


Figure 19

Example 7 – *Solution*

We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components. From Figure 20 we see that

5 $\mathbf{T}_1 = -|\mathbf{T}_1| \cos 50° \, \mathbf{i} + |\mathbf{T}_1| \sin 50° \, \mathbf{j}$

6 $\mathbf{T}_2 = |\mathbf{T}_2| \cos 32° \, \mathbf{i} + |\mathbf{T}_2| \sin 32° \, \mathbf{j}$

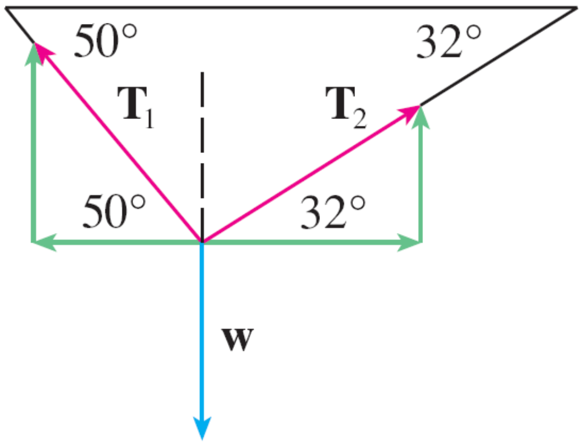


Figure 20

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100 \, \mathbf{j}$$

Example 7 – *Solution*

cont'd

Thus

$$(-|\mathbf{T}_1|\cos 50^\circ + |\mathbf{T}_2|\cos 32^\circ)\,\mathbf{i} + (|\mathbf{T}_1|\sin 50^\circ + |\mathbf{T}_2|\sin 32^\circ)\,\mathbf{j} = 100\,\mathbf{j}$$

Equating components, we get

$$-|\mathbf{T}_1| \cos 50^\circ + |\mathbf{T}_2| \cos 32^\circ = 0$$

$$|\mathbf{T}_1| \sin 50^\circ + |\mathbf{T}_2| \sin 32^\circ = 100$$

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1| \sin 50^\circ + \frac{|\mathbf{T}_1| \cos 50^\circ}{\cos 32^\circ} \sin 32^\circ = 100$$

Example 7 – *Solution*

cont'd

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50° + \tan 32° \cos 50°}$$

$$\approx 85.64 \text{ lb}$$

and

$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1| \cos 50°}{\cos 32°}$$

$$\approx 64.91 \text{ lb}$$

Substituting these values in (5) and (6), we obtain the tension vectors

$\mathbf{T}_1 \approx -55.05\mathbf{i} + 65.60\mathbf{j}$ $\quad\quad$ $\mathbf{T}_2 \approx 55.05\mathbf{i} + 34.40\mathbf{j}$