Vectors and the Geometry of Space

9

9.3 The Dot Product

The Dot Product

So far we have added two vectors and multiplied a vector by a scalar.

The question arises: Is it possible to multiply two vectors so that their product is a useful quantity?

One such product is the dot product, which we consider in this section.

Work and the Dot Product

Work and the Dot Product

An example of a situation in physics and engineering where we need to combine two vectors occurs in calculating the work done by a force.

We defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object.

Work and the Dot Product

Suppose, however, that the constant force is a vector **F** = $\overrightarrow{PR}$ pointing in some other direction, as in Figure 1.

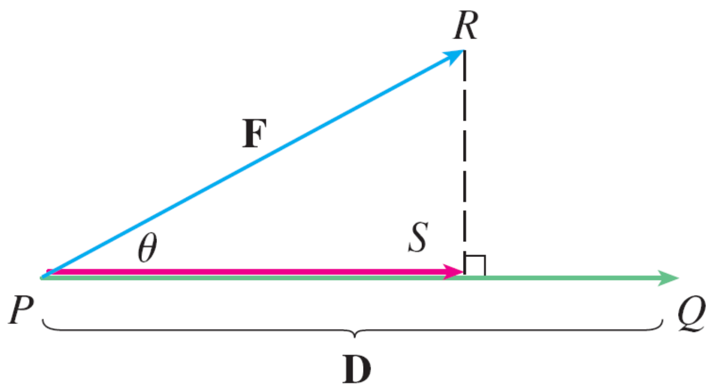

Figure 1

If the force moves the object from *P* to *Q*, then the **displacement vector** is **D** = $\overrightarrow{PQ}$. So here we have two vectors: the force **F** and the displacement **D**.

Work and the Dot Product

The **work** done by **F** is defined as the magnitude of the displacement, |**D**|, multiplied by the magnitude of the applied force in the direction of the motion, which, from Figure 1, is

$$|\overrightarrow{PS}| = |\mathbf{F}| \cos \theta$$

So the work done by **F** is defined to be

1 $$W = |\mathbf{D}|\,(|\mathbf{F}| \cos \theta) = |\mathbf{F}||\mathbf{D}| \cos \theta$$

Notice that work is a scalar quantity; it has no direction. But its value depends on the angle θ between the force and displacement vectors.

Work and the Dot Product

We use the expression in Equation 1 to define the dot product of two vectors even when they don't represent force or displacement.

Definition The **dot product** of two nonzero vectors **a** and **b** is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where θ is the angle between **a** and **b**, $0 \leqslant \theta \leqslant \pi$. (So θ is the smaller angle between the vectors when they are drawn with the same initial point.) If either **a** or **b** is **0**, we define $\mathbf{a} \cdot \mathbf{b} = 0$.

This product is called the **dot product** because of the dot in the notation **a** · **b**.

Work and the Dot Product

The result of computing $\mathbf{a} \cdot \mathbf{b}$ is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product**.

In the example of finding the work done by a force $\mathbf{F}$ in moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ by calculating $\mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos \theta$, it makes no sense for the angle θ between $\mathbf{F}$ and $\mathbf{D}$ to be $\pi/2$ or larger because movement from P to Q couldn't take place.

We make no such restriction in our general definition of $\mathbf{a} \cdot \mathbf{b}$, however, and allow θ to be any angle from 0 to π.

Example 1 – *Computing a Dot Product from Lengths and the Contained Angle*

If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution:

According to the definition,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/3)$$

$$= 4 \cdot 6 \cdot \frac{1}{2}$$

$$= 12$$

Work and the Dot Product

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$.

For such vectors we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/2)$$

$$= 0$$

and conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$.

The zero vector **0** is considered to be perpendicular to all vectors.

Work and the Dot Product

Therefore

2 Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Because $\cos \theta > 0$ if $0 \leq \theta < \pi/2$ and $\cos \theta < 0$ if $\pi/2 < \theta \leq \pi$, we see that $\mathbf{a} \cdot \mathbf{b}$ is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$.

We can think of $\mathbf{a} \cdot \mathbf{b}$ as measuring the extent to which **a** and **b** point in the same direction.

Work and the Dot Product

The dot product **a** · **b** is positive if **a** and **b** point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 3).

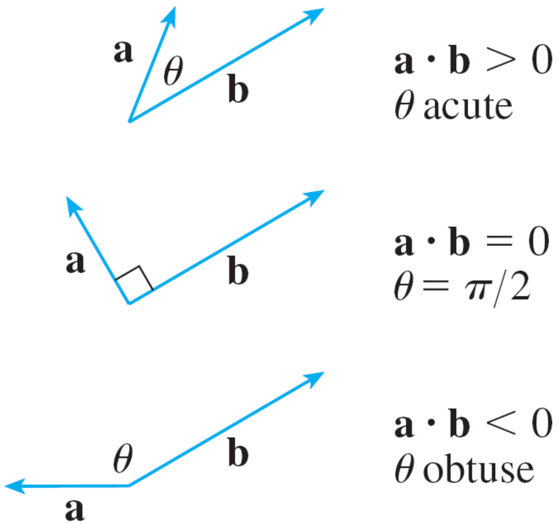

Figure 3

Work and the Dot Product

In the extreme case where **a** and **b** point in exactly the same direction, we have $\theta = 0$, so $\cos \theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

If **a** and **b** point in exactly opposite directions, then $\theta = \pi$ and so $\cos \theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

The Dot Product in Component Form

The Dot Product in Component Form

Suppose we are given two vectors in component form:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \qquad \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

We want to find a convenient expression for $\mathbf{a} \cdot \mathbf{b}$ in terms of these components. If we apply the Law of Cosines to the triangle in Figure 4, we get

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$

$$= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b}$$

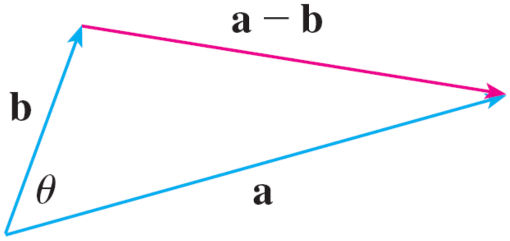

Figure 4

The Dot Product in Component Form

Solving for the dot product, we obtain

$$\mathbf{a} \cdot \mathbf{b} = \tfrac{1}{2}\,(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2)$$

$$= \tfrac{1}{2}\,[a_1{}^2 + a_2{}^2 + a_3{}^2 + b_1{}^2 + b_2{}^2 + b_3{}^2 - (a_1 - b_1)^2$$

$$- (a_2 - b_2)^2 - (a_3 - b_3)^2]$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The Dot Product in Component Form

Thus, to find the dot product of **a** and **b**, we multiply corresponding components and add.

The dot product of two-dimensional vectors is found in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

Example 3 – *Computing Dot Products from Components*

$$\langle 2, 4\rangle \cdot \langle 3, -1\rangle = 2(3) + 4(-1)$$

$$= 2$$

$$\langle -1, 7, 4\rangle \cdot \langle 6, 2, -\tfrac{1}{2}\rangle = (-1)(6) + 7(2) + 4(-\tfrac{1}{2})$$

$$= 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1)$$

$$= 7$$

The Dot Product in Component Form

The dot product obeys many of the laws that hold for ordinary products of real numbers.

These are stated in the following theorem.

Properties of the Dot Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

Projections

Projections

Figure 5 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point *P*. If *S* is the foot of the perpendicular from *R* to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$. (You can think of it as a shadow of **b**).

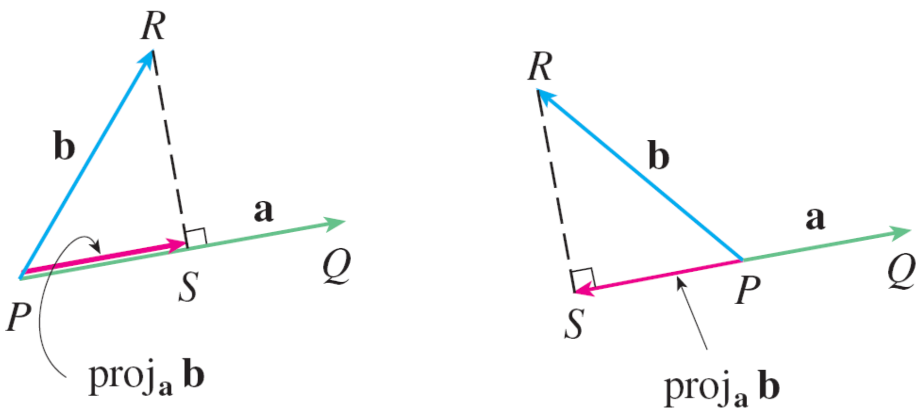

Figure 5

Vector projections

Projections

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}| \cos \theta$, where θ is the angle between **a** and **b**.
(See Figure 6.)

This is denoted by $\text{comp}_{\mathbf{a}}\, \mathbf{b}$.

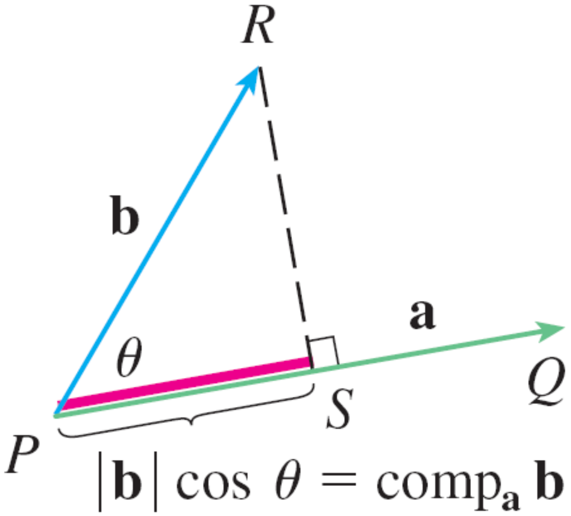

Figure 6

Scalar projection

Projections

Observe that it is negative if $\pi/2 < \theta \leq \pi$. (Note that we have used the component of the force **F** along the displacement **D**, $\text{comp}_{\mathbf{D}}\ \mathbf{F}$.)

The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

$$= |\mathbf{a}|(|\mathbf{b}| \cos \theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**.

Projections

Since

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**.

We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

Vector projection of **b** onto **a**: $$\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$$

Example 7

Find the scalar projection and vector projection of **b** = ⟨1, 1, 2⟩ onto **a** = ⟨–2, 3, 1⟩.

Solution:

Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

$$= \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}}$$

$$= \frac{3}{\sqrt{14}}$$

Example 7 – *Solution*

cont'd

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{3}{\sqrt{14}}\,\frac{\mathbf{a}}{|\mathbf{a}|}$$

$$= \frac{3}{14}\,\mathbf{a}$$

$$= \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$