# Vectors and the Geometry of Space

9

# 9.4 The Cross Product

# The Cross Product

The **cross product a** × **b** of two vectors **a** and **b**, unlike the dot product, is a vector. For this reason it is also called the **vector product**.

We will see that **a** × **b** is useful in geometry because it is perpendicular to both **a** and **b**.

But we introduce this product by looking at a situation where it arises in physics and engineering.

# Torque and the Cross Product

# Torque and the Cross Product

If we tighten a bolt by applying a force to a wrench as in Figure 1, we produce a turning effect called a *torque* $\boldsymbol{\tau}$.

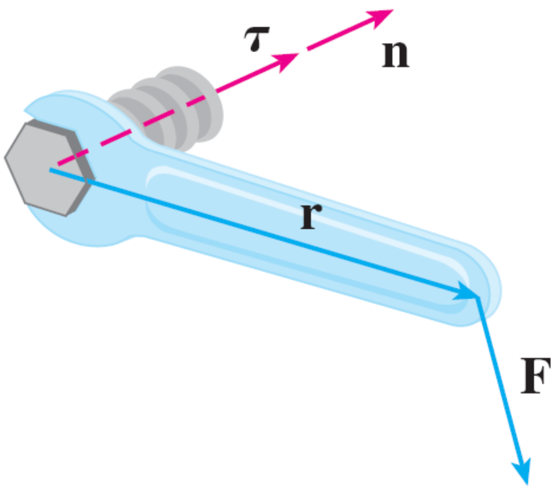


**Figure 1**

# Torque and the Cross Product

The magnitude of the torque depends on two things:

- The distance from the axis of the bolt to the point where the force is applied. This is |**r**|, the length of the position vector **r**.
- The scalar component of the force **F** in the direction perpendicular to **r**.

# Torque and the Cross Product

This is the only component that can cause a rotation and, from Figure 2, we see that it is

$$|\mathbf{F}| \sin \theta$$

where $\theta$ is the angle between the vectors **r** and **F**.

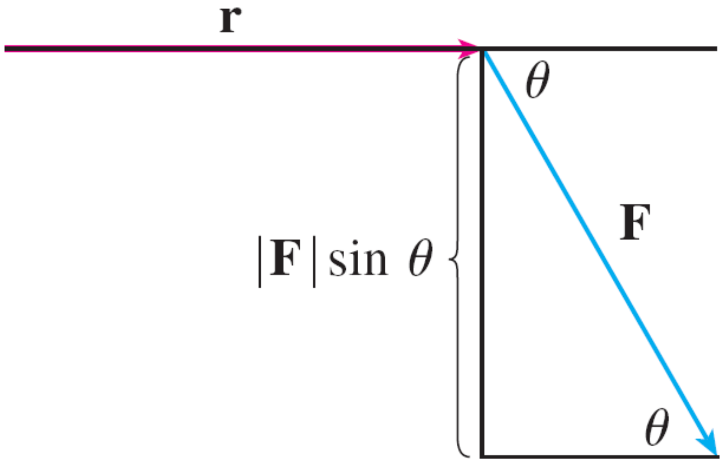


**Figure 2**

We define the magnitude of the torque vector to be the product of these two factors:

$$|\boldsymbol{\tau}| = |\mathbf{r}||\mathbf{F}| \sin \theta$$

# Torque and the Cross Product

The direction is along the axis of rotation. If **n** is a unit vector that points in the direction in which a right-threaded bolt moves (see Figure 1), we define the **torque** to be the vector

1 $$\boldsymbol{\tau} = (|\mathbf{r}||\mathbf{F}| \sin \theta)\mathbf{n}$$

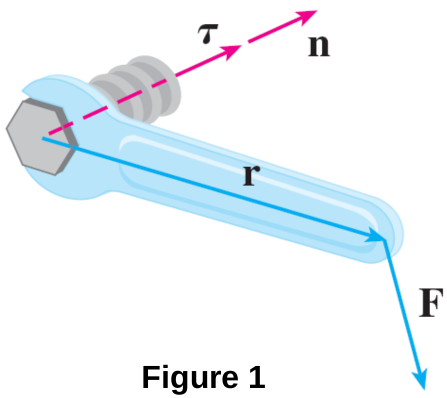

Figure 1

We denote this torque vector by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ and we call it the *cross product* or *vector product* of **r** and **F**.

# Torque and the Cross Product

The type of expression in Equation 1 occurs so frequently in the study of fluid flow, planetary motion, and other areas of physics and engineering, that we define and study the cross product of *any* pair of three-dimensional vectors **a** and **b**.

# Torque and the Cross Product

**Definition** If **a** and **b** are nonzero three-dimensional vectors, the **cross product** of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin \theta)\, \mathbf{n}$$

where $\theta$ is the angle between **a** and **b**, $0 \leqslant \theta \leqslant \pi$, and **n** is a unit vector perpendicular to both **a** and **b** and whose direction is given by the **right-hand rule**: If the fingers of your right hand curl through the angle $\theta$ from **a** to **b**, then your thumb points in the direction of **n**. (See Figure 3.)

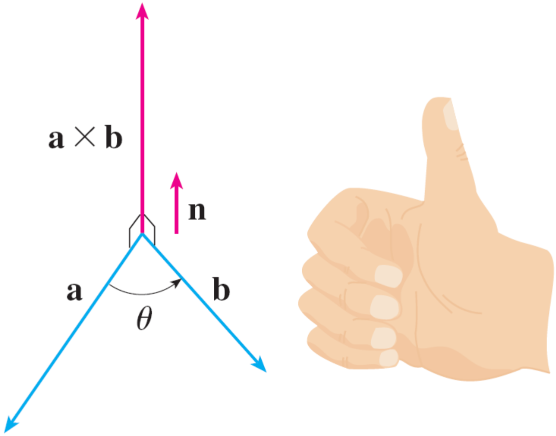


**Figure 3**

The right-hand rule gives the direction of **a x b**.

# Torque and the Cross Product

If either **a** or **b** is **0**, then we define **a** × **b** to be **0**. Because **a** × **b** is a scalar multiple of **n**, it has the same direction as **n** and so

> **a** × **b** is orthogonal to both **a** and **b**.

Notice that two nonzero vectors **a** and **b** are parallel if and only if the angle between them is 0 or $\pi$. In either case, $\sin\theta = 0$ and so **a** × **b** = **0**.

> Two nonzero vectors **a** and **b** are parallel if and only if **a** × **b** = **0**.

This makes sense in the torque interpretation: If we pull or push the wrench in the direction of its handle (so **F** is parallel to **r**), we produce no torque.

# Example 1

A bolt is tightened by applying a 40-N force to a 0.25-m wrench, as shown in Figure 4. Find the magnitude of the torque about the center of the bolt.

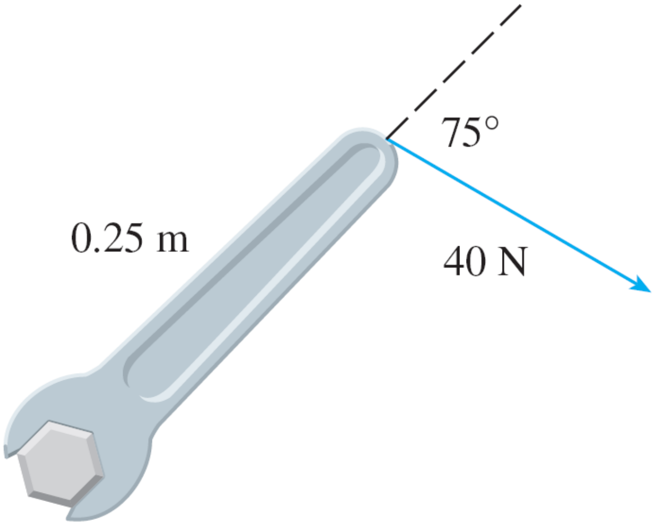


**Figure 4**

# Example 1 – *Solution*

The magnitude of the torque vector is

$$
\begin{aligned}
|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| &= |\mathbf{r}||\mathbf{F}| \sin 75^\circ \, |\mathbf{n}| \\
&= (0.25)(40) \sin 75^\circ \\
&= 10 \sin 75^\circ \\
&\approx 9.66 \text{ N} \cdot \text{m}
\end{aligned}
$$

If the bolt is right-threaded, then the torque vector itself is

$$
\begin{aligned}
\boldsymbol{\tau} &= |\boldsymbol{\tau}| \, \mathbf{n} \\
&\approx 9.66 \, \mathbf{n}
\end{aligned}
$$

where **n** is a unit vector directed down into the page.

# Torque and the Cross Product

In general, the right-hand rule shows that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Another algebraic law that fails for the cross product is the associative law for multiplication; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

For instance, if $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i}$, and $\mathbf{c} = \mathbf{j}$, then

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

whereas

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

# Torque and the Cross Product

However, some of the usual laws of algebra *do* hold for cross products:

**Properties of the Cross Product** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and $c$ is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

# Torque and the Cross Product

A geometric interpretation of the length of the cross product can be seen by looking at Figure 6.

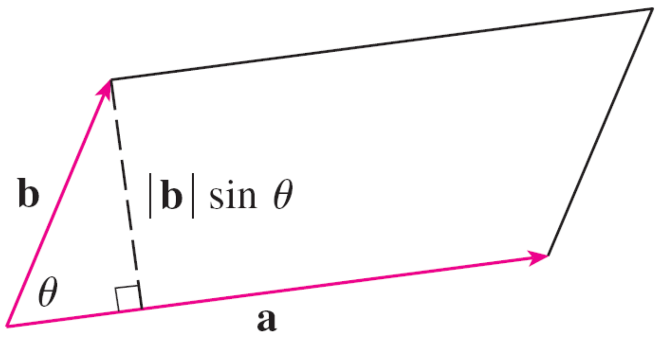


Figure 6

If **a** and **b** are represented by directed line segments with the same initial point, then they determine a parallelogram with base |**a**|, altitude |**b**| sin $\theta$, and area

$$A = |\mathbf{a}|(|\mathbf{b}| \sin \theta) = |\mathbf{a} \times \mathbf{b}|$$

The length of the cross product **a** × **b** is equal to the area of the parallelogram determined by **a** and **b**.

# The Cross Product in Component Form

# The Cross Product in Component Form

Suppose **a** and **b** are given in component form:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

We can express **a** × **b** in component form by using the Vector Distributive Laws together with the results from Example 2:

$$\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

$$= a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k}$$

$$+ a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k}$$

$$+ a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k}$$

# The Cross Product in Component Form

$$= a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} + a_3b_2(-\mathbf{i})$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

**2** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

In order to make this expression for $\mathbf{a} \times \mathbf{b}$ easier to remember, we use the notation of determinants.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

# The Cross Product in Component Form

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{3}$$

# The Cross Product in Component Form

Observe that each term on the right side of Equation 3 involves a number $a_i$ in the first row of the determinant, and $a_i$ is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which $a_i$ appears.

Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38$$

# The Cross Product in Component Form

If we now rewrite the expression for **a** × **b** in (2) using second-order determinants and the standard basis vectors **i**, **j**, and **k**, we see that the cross product of the vectors **a** = $a_1$**i** + $a_2$**j** + $a_3$**k** and **b** = $b_1$**i** + $b_2$**j** + $b_3$**k** is

**4**
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

In view of the similarity between Equations 3 and 4, we often write

**5**
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

# The Cross Product in Component Form

Although the first row of the symbolic determinant in Equation 5 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 3, we obtain Equation 4.

The symbolic formula in Equation 5 is probably the easiest way of remembering and computing cross products.

# Example 3 – *Cross Product of Vectors in Component Form*

If **a** = ⟨1, 3, 4⟩ and **b** = ⟨2, 7, –5⟩, then, from Equation 5, we have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k}$$

$$= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

# Triple Products

# Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the **scalar triple product** of the vectors **a**, **b**, and **c**. Its geometric significance can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**. (See Figure 7.)

The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If $\theta$ is the angle between the vectors **a** and $\mathbf{b} \times \mathbf{c}$, then the height $h$ of the parallelepiped is $h = |\mathbf{a}||\cos \theta|$. (We must use $|\cos \theta|$ instead of $\cos \theta$ in case $\theta > \pi/2$.) Thus the volume of the parallelepiped is

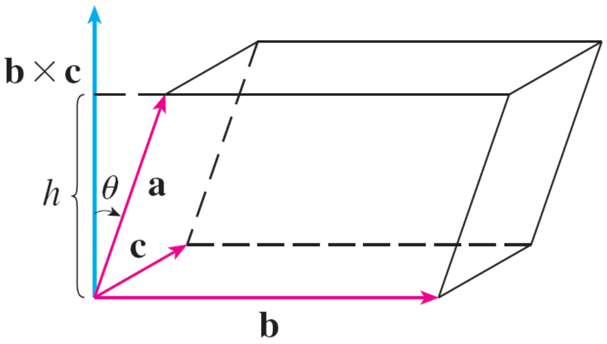


Figure 7

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos \theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

# Triple Products

Therefore we have proved the following:

The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Instead of thinking of the parallelepiped as having its base parallelogram determined by **b** and **c**, we can think of it with base parallelogram determined by **a** and **b**. In this way, we see that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

# Triple Products

But the dot product is commutative, so we can write

**6** $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

Suppose that **a**, **b**, and **c** are given in component form:

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \quad \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

Then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \left[ \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \right]$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

# Triple Products

This shows that we can write the scalar triple product of **a**, **b**, and **c** as the determinant whose rows are the components of these vectors:

7

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

# Example 6 – *Coplanar Vectors*

Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar; that is, they lie in the same plane.

Solution:

We use Equation 7 to compute their scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

# Example 6 – *Solution*

cont'd

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18)$$

$$= 0$$

Therefore the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar.

# Triple Products

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is called the **vector triple product** of **a**, **b**, and **c**.

$$\boxed{8} \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$