MA 182 **An Integral Involving Powers of Sines and Cosines**
Section 5.7

Solutions By Dr. Katiraie

Evaluate the integral $\int \sin^3 x \cos^3 x \, dx$

Rewrite the integral as $\int \sin^2 x \cos^3 x \sin x \, dx$ and then use the identity $\sin^2\theta + \cos^2\theta = 1$ to convert $\sin^2 x \cos^3 x$ to powers of $\cos x$. Use a u-substitution with $u = \cos x, du = -\sin x \, dx$. You should then be able to integrate. Be sure to rewrite the answer in terms of the original variable. Your answer should involve powers of $\cos x$.

$$\int \sin^3 x \cos^3 x \, dx = \int \sin^2 x \cos^3 x \sin x \, dx$$

$$= \int (1-\cos^2 x)\cos^3 x \sin x \, dx$$

$$= \int (\cos^3 x - \cos^5 x) \sin x \, dx$$

$$= \int \cos^3 x \sin x \, dx - \int \cos^5 x \sin x \, dx$$

let $u = \cos x$ $\quad$ $u = \cos x$

$du = -\sin x \, dx$ $\quad$ $du = -\sin x \, dx$

$$= -\int u^3 \, du + \int u^5 \, du$$

$$= -\frac{u^4}{4} + \frac{u^6}{5} + c$$

$$= \boxed{-\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + c}$$

Prove that

$$\int \sin^3 x \cos^3 x \, dx = -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + c$$

$$\left(-\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + c\right)'$$

$$= -\frac{4\cos^3 x}{4}(-\sin x) + \frac{6\cos^5 x}{6}(-\sin x) + 0$$

$$= \cos^3 x \sin x - \cos^5 x \sin x \; ; \left(\text{Factor out } \cos^3 x \sin x\right)$$

$$= \cos^3 x \sin x \left(1 - \cos^2 x\right) \quad \text{and } \underline{\underline{\text{Recall}}} \quad \sin^2 x + \cos^2 x = 1$$

then $\sin^2 x = 1 - \cos^2 x$

$$= \cos^3 x \sin x (\sin^2 x)$$

$$\boxed{= \cos^3 x \sin^3 x}$$

Section 5.7 Worksheet Solutions

#1) $\int \sin^3 2x \, dx$

$\int \sin^2(2x) \sin(2x) \, dx$

$\int \left(1-\cos^2(2x)\right) \sin(2x) \, dx$

$= \int \sin(2x) \, dx - \int \cos^2(2x) \sin(2x) \, dx$

let $u = 2x$

$du = 2 \, dx$

$u = \cos(2x)$

$du = -\sin(2x) \cdot 2 \, dx$

$= \frac{1}{2} \int \sin(2x) \, 2 \, dx + \frac{1}{2} \int \cos^2(2x) \left(-2\sin(2x) \, dx\right)$

$= \frac{1}{2} \int \sin(u) \, du + \frac{1}{2} \int u^2 \, du$

$= -\frac{1}{2} \cos u + \frac{1}{2} \frac{u^3}{3} + c$

Recall $u = 2x$ | Recall $u = \cos(2x)$

$$\boxed{= -\frac{1}{2} \cos(2x) + \frac{1}{6} \cos^3(2x) + c}$$

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Note:

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

1b) $\int \cos^2(5x)\,dx$

$= \int \frac{1}{2}(1 + \cos 10x)\,dx$

$= \int \frac{1}{2}\,dx + \int \frac{1}{2}\cos(10x)\,dx$

let $u = 10x$

$du = 10\,dx$

$= \int \frac{1}{2}\,dx + \frac{1}{2}\frac{1}{10}\int \cos(10x)\,10\,dx$

$= \int \frac{1}{2}\,dx + \frac{1}{20}\int \cos(u)\,du$

$= \frac{1}{2}x + \frac{1}{20}\sin(u) + C$

$\boxed{= \frac{1}{2}x + \frac{1}{20}\sin(10x) + C}$

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#1 c) $\int \tan^6 x \sec^4 x \, dx$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\int \tan^6 x \sec^2 x (\tan^2 x + 1) \, dx$

$\int \tan^8 x \sec^2 x \, dx + \int \tan^6 x \sec^2 x \, dx$

let $u = \tan x$

$du = \sec^2 x \, dx$

$= \int u^8 \, du + \int u^6 \, du$

$= \frac{u^9}{9} + \frac{u^7}{7} + c$

$$= \boxed{\frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + c}$$

#1)

(d) $\int \tan^3 x \sec^3 x \, dx$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\frac{}{\cos}$: $\tan^2 x = \sec^2 x - 1$

$= \int \tan^2 x \tan x \sec^3 x \, dx$

$= \int (\sec^2 x - 1) \tan x \sec^3 x \, dx$

$= \int \sec^5 x \tan x \, dx - \int \sec^3 x \tan x \, dx$

$= \int \sec^4 x \sec x \tan x \, dx - \int \sec^2 x \sec x \tan x \, dx$

let $u = \sec x$; $du = \sec x \tan x \, dx$

$= \int u^4 \, du - \int u^2 \, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C$

$$\boxed{= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

#2 $\int_0^1 \cos^4 2x \sin 2x \, dx$

let $u = \cos(2x)$

$du = -\sin(2x) \cdot 2 \, dx$

$= -\frac{1}{2} \int \cos^4(2x) \left(-2 \sin(2x) dx\right)$

$= -\frac{1}{2} \int u^4 \, du$

$= -\frac{1}{2} \frac{u^5}{5}$

$= -\frac{1}{10} \cos^5(2x) \Big|_0^1$

$= -\frac{1}{10} \left(\cos^5(2) - \cos^5(0)\right)$ Mode: Radian

$\cong \boxed{0.10125}$

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#3

$$v(t) = \cos^2(\pi t)$$

$$\int_0^5 \cos^2(\pi t)\,dt \qquad \text{Recall } \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

$$= \int_0^5 \tfrac{1}{2}(1+\cos 2\pi t)\,dt$$

$$= \int_0^5 \tfrac{1}{2}\,dt + \int_0^5 \tfrac{1}{2}\cos(2\pi t)\,dt$$

$$u = 2\pi t \qquad du = 2\pi\,dt$$

$$= \tfrac{1}{2}t + \tfrac{1}{2}\cdot\tfrac{1}{2\pi}\int_0^5 \cos(2\pi t)\;2\pi\,dt$$

$$= \tfrac{1}{2}t + \tfrac{1}{4\pi}\int_0^5 \cos(u)\,du$$

$$= \tfrac{1}{2}t + \tfrac{1}{4\pi}\sin u$$

$$= \tfrac{1}{2}t + \tfrac{1}{4\pi}\sin(2\pi t)\Big|_0^5$$

$$= \left(\tfrac{1}{2}(5) + \underbrace{\tfrac{1}{4\pi}\sin(10\pi)}_{0}\right) - \underbrace{\left(\tfrac{1}{2}(0) + \tfrac{1}{4\pi}\sin(0)\right)}_{0} = \boxed{\tfrac{5}{2}\text{ feet}}$$

Section 5.7 Review

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{x^2+1}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{\frac{1}{a}}{\sqrt{\frac{a^2}{a^2}-\left(\frac{x}{a}\right)^2}}\,dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,\frac{1}{a}dx$$

$$= \boxed{\sin^{-1}\left(\frac{x}{a}\right)+c}$$

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{\frac{1}{a^2}}{\frac{a^2}{a^2}+\frac{x^2}{a^2}}\,dx = \int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,\frac{1}{a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

Example(1) $$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{\frac{1}{3}}{\sqrt{\frac{9}{9}-\left(\frac{x}{3}\right)^2}}\,dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\,\frac{1}{3}dx$$

$$= \boxed{\sin^{-1}\left(\frac{x}{3}\right)+c}$$

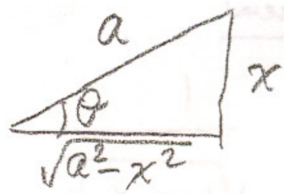
Example(2) $$\int \frac{1}{x^2+2}\,dx = \int \frac{\frac{1}{2}}{\frac{x^2}{2}+\frac{2}{2}}\,dx$$

$$= \int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,\frac{1}{2}\,dx = \int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\frac{1}{\sqrt{2}}dx = \boxed{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c}$$

See 8.1 WU

$\sqrt{a^2-x^2}$ let $\theta = \sin^{-1}\left(\frac{x}{a}\right)$



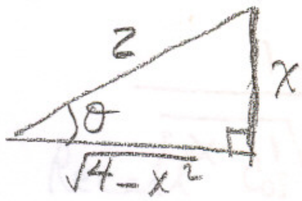
$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx$$

let $\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\sin\theta = \frac{x}{2}$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$



$$\int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\, d\theta$$

$$\int \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}}\,2\cos\theta\, d\theta$$

$$= \int \frac{4\sin^2\theta\ \cancel{2\cos\theta}\, d\theta}{\cancel{2\cos\theta}}$$

Recall $\sin^2\theta = \frac{1}{2}\left(1-\cos(2\theta)\right)$

$$= 4\int \frac{1}{2}\left(1-\cos(2\theta)\right)d\theta$$

$$= 2\int \left(1-\cos(2\theta)\right)d\theta$$

$$= 2\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C = 2\theta - \sin(2\theta) + C = 2\theta - 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\,\frac{x}{2}\,\frac{\sqrt{4-x^2}}{2} + C$$

$$\boxed{= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C}$$

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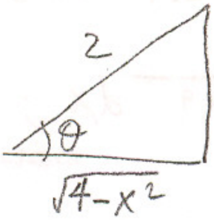
$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx \quad \text{let } \theta = \sin^{-1}\left(\frac{x}{2}\right)$$



$$\sin\theta = \frac{x}{2}$$

$$x = 2\sin\theta$$

$$dx = 2\cos\theta\, d\theta$$

$$\cos^2 x = \frac{1}{2}(1+\cos(2x))$$

$$\sin^2 x = \frac{1}{2}(1-\cos(2x))$$

$$\int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta$$

$$\int \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}}\,2\cos\theta\,d\theta = \int \frac{4\sin^2\theta}{\cancel{\sqrt{4\cos^2\theta}}}\,\cancel{2\cos\theta}\,d\theta$$

$$\int 4\sin^2\theta\,d\theta = \int 4\left(\frac{1}{2}(1-\cos(2\theta))\right)d\theta$$

$$= \int \left(2 - 2\cos(2\theta)\right)d\theta$$

$$= 2\theta - \sin(2\theta) + c$$

$$= 2\theta - 2\sin\theta\cos\theta + c$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\,\frac{x}{\cancel{2}}\cdot\frac{\sqrt{4-x^2}}{\cancel{2}} + c$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + c$$

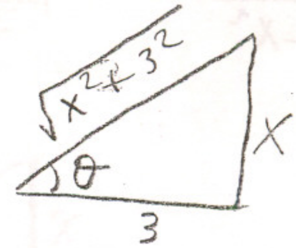
Section 5.7 W.S

$$\int x^3 \sqrt{x^2+9}\, dx \; ; \; \text{let } \theta = \tan^{-1}\left(\frac{x}{3}\right)$$

$$\tan\theta = \frac{x}{3}$$

$$x = 3\tan\theta$$

$$dx = 3\sec^2\theta\, d\theta$$



$$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9} = \sqrt{9\tan^2\theta+9}$$

$$= \sqrt{9(\tan^2\theta+1)}$$

$$= \sqrt{9\sec^2\theta} = 3\sec\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\int x^3\sqrt{x^2+9}\, dx = \int 27\tan^3\theta \;\; 3\sec\theta \;\; 3\sec^2\theta\, d\theta$$

$$= 243\int \tan^3\theta\sec^3\theta\, d\theta$$

$$= 243\int (\sec^2\theta - 1)\tan\theta\sec^3\theta\, d\theta$$

$$= 243\left[\int \sec^5\theta\tan\theta\, d\theta - \int \sec^3\theta\tan\theta\, d\theta\right]$$

$$= 243\left[\int \sec^4\theta\sec\theta\tan\theta\, d\theta - \int \sec^2\theta\sec\theta\tan\theta\, d\theta\right]$$

$$= 243\left[\frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3}\right] + c$$

$$= \frac{243}{5}\left(\frac{\sqrt{x^2+9}}{3}\right)^5 - \frac{243}{3}\left(\frac{\sqrt{x^2+9}}{3}\right)^3 + c$$

$$\boxed{= \frac{1}{5}(x^2+9)^{\frac{5}{2}} - 3(x^2+9)^{\frac{3}{2}} + c}$$