Section 4.5

Using Exponential Functions to Model Data

Definition

An **exponential model** is an exponential function, or its graph, that describes the relationship between two quantities for an authentic situation. If all of the data points for a situation lie on an exponential curve, then we say that the independent and dependent variables are **exponentially related.**

(Continues)

Finding an Equation of an Exponential Curve

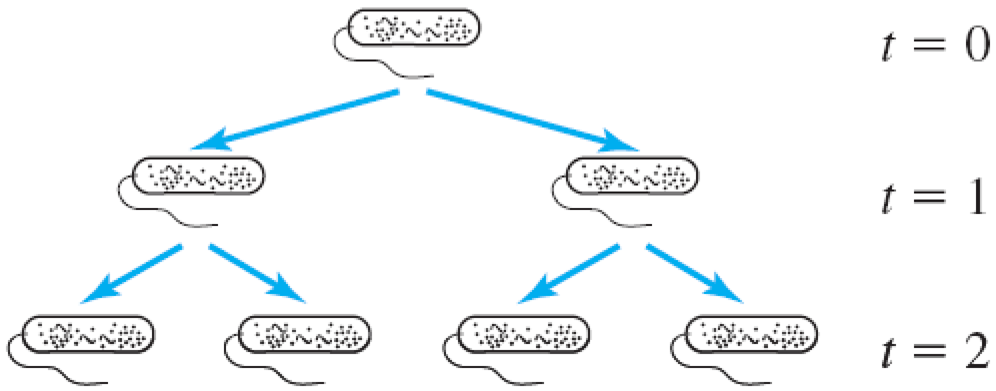
Using the Base Multiplier Property to Find Exponential Functions

Definition Continued

If no exponential curve contains all of the data points, but an exponential curve comes close to all of the data points (and perhaps contains some of them), then we say that the variables are **approximately exponentially related.**

Example

Suppose that a peach has 3 million bacteria on it at noon on Monday and that one bacterium divides into two bacteria every hour, on average.



Let B be the number of bacteria (in millions) on the peach at t hours after noon on Monday.

Example Continued

1. Find an equation of f

Solution

- Create table with assumptions
- As t increases by 1, B changes by multiplying by 2
- So, model is of the form $f(t) = a(2)^t$

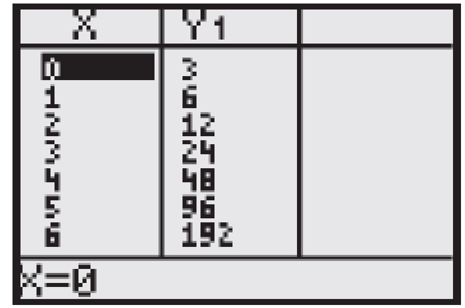
t (hours)	$B = f(t)$ (millions)
0	3
1	6
2	12
3	24
4	48

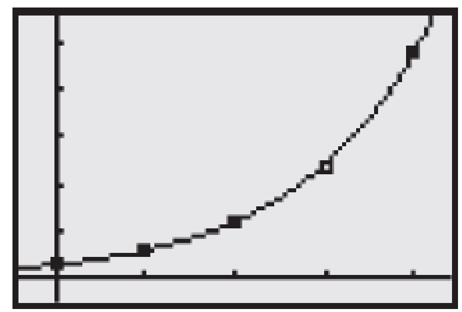
Exponential Model, Exponential Related, Approximately Exponentially Related

Using Base Multiplier Property to Find a Model

Solution Continued

- *B*-intercept is (0, 3), so $f(t) = 3(2)^t$
- Verify solution using graphing calculator





Example Continued

2. Predict the number of bacteria on the peach at noon on Tuesday.

Solution

- Use $t = 24$ to represent noon on Tuesday
- Substitute 24 for t

$$f(24) = 3(2)^{24} = 50{,}331{,}648$$

- According to the model there would be 50,331,648 million bacteria
- Omit "million": 50,331,648,000,000
- That's about 50 trillion bacteria at noon on Tuesday

Modeling with an Exponential Function

Using Base Multiplier Property to Find a Model

Summary

- Exponential function $y = ab^t$ the y-intercept is $(0, a)$
- y is the quantity at time t
- a is the quantity at time $t = 0$

Example

A person invests \$5000 in an account that earns 6% interest compounded annually.

1. Let $V = f(t)$ be the value of the account at t years after the money is invested. Find an equation of f.

Modeling with an Exponential Function

Using Base Multiplier Property to Find a Model

Solution

- Each year the investment value is the previous value (100%) plus 6% of the previous value
- Each year the investment is 106% of the previous year
- As t increases by 1, the vale of V is multiplied by 1.06: $b = 1.06$
- Value of account starts at $5000: $a = 5000$
- $f(t) = 5000(1.06)^t$

t	$V = f(t)$
0	5000.00
1	$5000.00(1.06) = 5300.00$
2	$5300.00(1.06) = 5618.00$
3	$5618.00(1.06) = 5955.08$
4	$5955.08(1.06) \approx 6312.38$

Modeling with an Exponential Function

Using Base Multiplier Property to Find a Model

Example Continued

2. What will be the value after 10 years?

Solution

- Substitute 10 for *t*

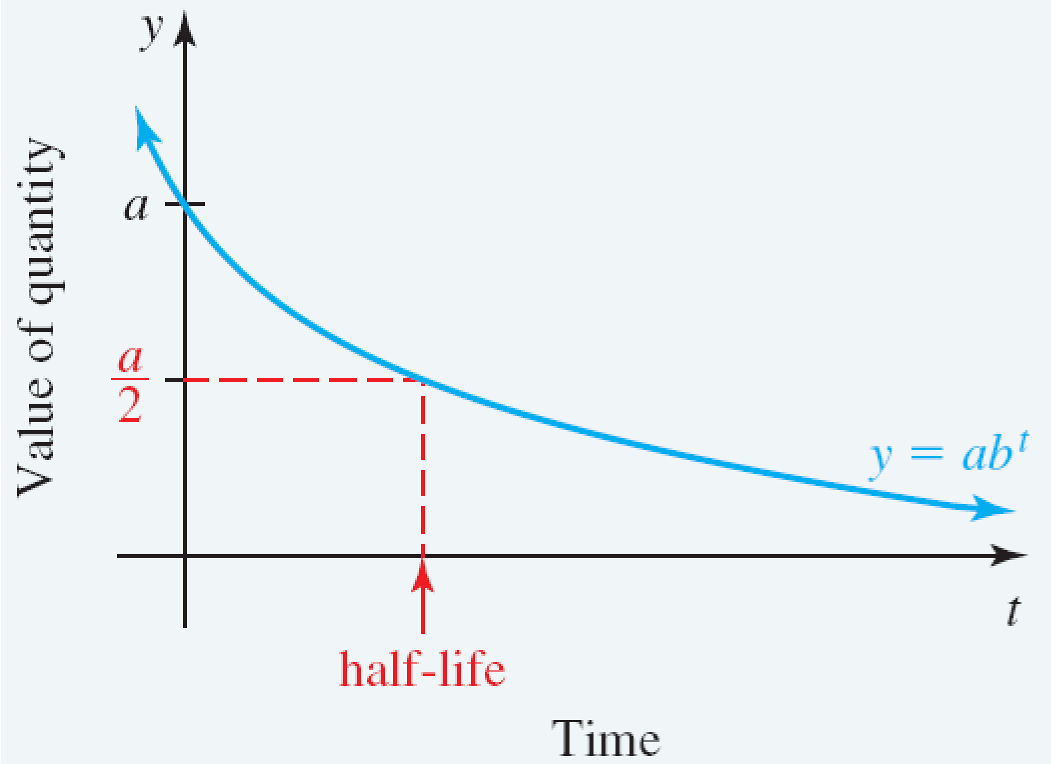
$$f(10) = 5000(1.06)^{10} \approx 8954.24$$

- Value will be $8954.24 in 10 years

Definition: Half-life

Definition

If a quantity decays exponentially, the **half-life** is the amount of time it takes for that quantity to be reduces to half.



Definition: Half-life

Example

The world's worst nuclear accident occurred in Chernobyl, Ukraine, on April 26, 1986. Immediately afterward, 28 people died from acute radiation sickness. So far, about 25,000 people have died from exposure to radiation, mostly due to the release of the radioactive element cesium-137 (Source: *Medicine Worldwide)*. Cesium-137 has a half-life of 30 years. Let $P = f(t)$ be the percent of the cesium-137 that remains at t years since 1986.

Example Continued

1. Find an equation of f.

Solution

Method 1

- At t = 0, 100% of the cesium-137 is present
- At t = 30, ½(100) = 50 percent is present
- At t = 60, ½(½)(100) = 25 percent
- Situation can be modeled using exponential function

Modeling with an Exponential Function

Half-life Applications

Solution Continued

- Each exponent in second column is equal to the value of t in the first column divided by 30
- Equation of f is:

$$f(t) = 100\left(\frac{1}{2}\right)^{t/30}$$

Year t	Percent P
0	$100 = 100\left(\frac{1}{2}\right)^0$
30	$100 \cdot \frac{1}{2} = 100 \cdot \left(\frac{1}{2}\right)^1$
60	$100 \cdot \frac{1}{2} \cdot \frac{1}{2} = 100\left(\frac{1}{2}\right)^2$
90	$100 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 100\left(\frac{1}{2}\right)^3$
t	$100\left(\frac{1}{2}\right)^{t/30}$

Solution Continued

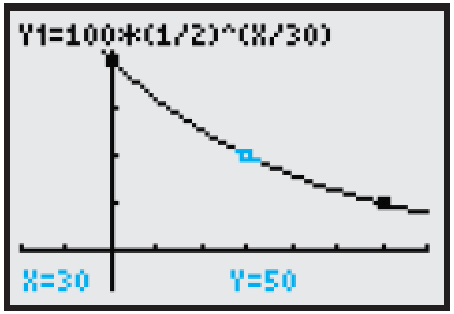
- Use a graphing calculator table and graph to verify solution

$$f\left(t\right)=100\left(\frac{1}{2}\right)^{t/30}=100\left(\frac{1}{2}\right)^{\frac{1}{30}\cdot t}=100\left(\left(\frac{1}{2}\right)^{\frac{1}{30}}\right)^{t}$$

- $\left(\frac{1}{2}\right)^{1/30}\approx 0.977$ can be written $f\left(t\right)=100\left(0.977\right)^{t}$

X	Y1
0	100
30	50
60	25
90	12.5
120	6.25
150	3.125
180	1.5625

X=0



Solution Continued

Method 2

- We know the points (0, 100) and (30, 50)
- P-intercept is (0, 100), so $a = 100$

$$P = f(t) = 100b^t$$

- Substitute (30, 50)

$$50 = 100b^{30} \quad \text{Substitute 30 for } t \text{ and 50 for } f(t).$$

$$100b^{30} = 50 \quad \text{If } c = d\text{, then } d = c.$$

$$b^{30} = \frac{50}{100} \quad \text{Divide both sides by 100.}$$

Modeling with an Exponential Function

Half-life Applications

Solution Continued

$$b^{30} = \frac{1}{2}$$ Simplify.

$$b = \pm\left(\frac{1}{2}\right)^{1/30}$$ The solution of $b^{30} = k$ is $\pm k^{1/30}$.

$$\approx 0.977$$ Compute; base of an exponential function is positive.

- Equation is the same $f(t) = 100\left(\frac{1}{2}\right)^{t/30}$

Example Continued

2. Describe the meaning of the base of *f*.

Solution

- Base of *f* is 0.977
- Each year 97.7% of the previous year's cesium-137 is present
- Or, cesium-137 decays by 2.3% per year

Example Continued

3. What percent of the cesium-137 will remain in 2010?

Solution

- 2010 – 1986 = 24
- Substitute 24 for *t*

$$f(24) = 100(0.977)^{24} \approx 57.21$$

- In 2010, about 57.2% of the cesium-137 will remain

Summary

If $f(t) = ab^t$, where $a > 0$, models a quantity at time t, then the percentage of change is constant. In particular,

- If $b > 1$, then the quantity grows exponentially at a rate of $b - 1$ percent (in decimal form) per unit of time.
- If $0 < b < 1$, then the quantity decays exponentially at a rate of $1 - \mathrm{b}$ percent (in decimal form) per unit of time.

Modeling with an Exponential Function

Finding a Model by Using Data Described in Words

Example

Sales of energy and nutrition bars have grown approximately exponentially from $0.2 billion in 1997 to $1.2 billion in 2004 (Source: *Frost & Sullivan)*. Predict the sales in 2011.

Solution

- Let s be the sales (in billions of dollars) of energy and nutrition bars
- Let t be the years since 1997

Modeling with an Exponential Function

Finding a Model by Using Data Described in Words

Solution Continued

- Create a table
- t and s are approx. exponential
- We want $s = ab^t$
- S-intercept is (0, 0.2)

Years Since 1997 t	Sales (billions of dollars) s
0	0.2
7	1.2

$$s = 0.2b^t$$

- Substitute (7, 1.2) and solve for b

Solution Continued

$1.2 = 0.2b^7$	Substitute 7 for t and 1.2 for s.
$0.2b^7 = 1.2$	If $c = d$, then $d = c$.
$b^7 = 6$	Divide both sides by 0.2.
$b = 6^{1/7}$	The solution of $b^7 = k$ is $k^{1/7}$.
$b \approx 1.292$	Compute.

- Substitute 1.292 for b

$$s = 0.2(1.292)^t$$

- 2011 – 1997 = 14, so substitute 14 for t

Modeling with an Exponential Function

Finding a Model by Using Data Described in Words

Solution Continued

$$s = 0.2(1.292)^{14} \approx 7.22$$

- Model predicts that sales will reach $7.22 billion in 2011
- Verify work on graphing calculator