Section 5.5

Properties of Logarithms

1 Work with the Properties of Logarithms

EXAMPLE

Establishing Properties of Logarithms

(a) Show that $\log_a 1 = 0$.

(b) Show that $\log_a a = 1$.

$$\log_a 1 = 0 \qquad \log_a a = 1$$

THEOREM

Properties of Logarithms

In the properties given next, M and a are positive real numbers, $a \neq 1$, and r is any real number.

The number $\log_a M$ is the exponent to which a must be raised to obtain M. That is,

$$a^{\log_a M} = M \qquad \textbf{(1)}$$

The logarithm to the base a of a raised to a power equals that power. That is,

$$\log_a a^r = r \qquad \textbf{(2)}$$

EXAMPLE **Using Properties (1) and (2)**

(a) $\log_{\pi} \pi^{3} = 3$ Property (2)

(b) $5^{\log_5 \sqrt{3}} = \sqrt{3}$ Property (1)

(c) $\ln e^{0.35t} = 0.035t$ Property (2)

THEOREM

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, with $a \neq 1$, and r is any real number.

The Log of a Product Equals the Sum of the Logs

$$\log_a(MN) = \log_a M + \log_a N \qquad \textbf{(3)}$$

The Log of a Quotient Equals the Difference of the Logs

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N \qquad \textbf{(4)}$$

The Log of a Power Equals the Product of the Power and the Log

$$\log_a M^r = r \log_a M \qquad \textbf{(5)}$$

2 Write a Logarithmic Expression as a Sum or Difference of Logarithms

EXAMPLE

Writing a Logarithmic Expression as a Sum of Logarithms

Write $\log_2\left(x^2\sqrt[3]{x-1}\right)$, $x>1$, as a sum of logarithms.

Express all powers as factors.

$$\log_2 x^2 + \log_2 \sqrt[3]{x-1}$$

$$\log_a(MN) = \log_a M + \log_a N$$

$$= \log_2 x^2 + \log_2 (x-1)^{\frac{1}{3}}$$

$$= 2\log_2 x + \frac{1}{3}\log_2 (x-1)$$

$$\log_a M^r = r\log_a M$$

EXAMPLE

Writing a Logarithmic Expression as a Difference of Logarithms

Write $\log_6 \dfrac{x^4}{\left(x^2+3\right)^2}$, $x \neq 0$, as a difference of logarithms.

Express all powers as factors.

$$\log_6 x^4 - \log_6 \left(x^2+3\right)^2$$

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$$

$$4\log_6 x - 2\log_6 \left(x^2+3\right)$$

$$\log_a M^r = r\log_a M$$

EXAMPLE

Writing a Logarithmic Expression as a Sum and Difference of Logarithms

Write $\ln \frac{x^3\sqrt{x-2}}{(x+1)^2}$, $x > 2$, as a sum and difference of logarithms.

Express all powers as factors.

$$\ln x^3\sqrt{x-2} - \ln(x+1)^2$$

$$\ln x^3 + \ln(x-2)^{\frac{1}{2}} - \ln(x+1)^2$$

$$3\ln x + \frac{1}{2}\ln(x-2) - 2\ln(x+1)$$

3 Write a Logarithmic Expression as a Single Logarithm

EXAMPLE **Writing Expressions as a Single Logarithm**

Write each of the following as a single logarithm.

(a) $3\ln 2+\ln\left(x^2\right)=\ln 2^3+\ln\left(x^2\right)\quad=\ln\left(8x^2\right)$

Property (5) Property (3)

(b) $\frac{1}{2}\log_a 4-2\log_a 5=\log_a 4^{\frac{1}{2}}-\log_a 5^2=\log_a\left(\frac{2}{25}\right)$

Property (5)

Property (4)

(c) $-2\log_a 3+3\log_a 2-\log_a\left(x^2+1\right)$

$=\log_a 3^{-2}+\log_a 2^3-\log_a\left(x^2+1\right)\quad=\log_a\left(2^3\left(3^{-2}\right)\right)-\log_a\left(x^2+1\right)$

Property (5)

$=\log_a\left(\frac{8}{9\left(x^2+1\right)}\right)$

Property (4)

THEOREM

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$.

$$\text{If } M = N \text{, then } \log_a M = \log_a N. \qquad \textbf{(7)}$$

$$\text{If } \log_a M = \log_a N \text{, then } M = N. \qquad \textbf{(8)}$$

4 Evaluate Logarithms Whose Base Is Neither 10 Nor *e*

EXAMPLE

Approximating a Logarithm Whose Base Is Neither 10 Nor e

Approximate $\log_3 12$. Round answer to four decimal places.

$$y = \log_3 12$$

$$3^y = 12$$ Exponential form

$$\ln 3^y = \ln 12$$ Property (7)

$$y \ln 3 = \ln 12$$ Property (5)

$$y = \frac{\ln 12}{\ln 3}$$ Exact value

$$y \approx 2.2619$$ Approximate value

THEOREM

Change-of-Base Formula

If $a \neq 1$, $b \neq 1$, and M are positive real numbers, then

$$\log_a M = \frac{\log_b M}{\log_b a}$$

$$\log_a M = \frac{\log M}{\log a} \quad \text{and} \quad \log_a M = \frac{\ln M}{\ln a}$$

EXAMPLE Using the Change-of-Base Formula

Approximate: (a) $\log_5 89$ (b) $\log_{\sqrt{2}} \sqrt{5}$

Round answers to four decimal places.

(a) $$\log_5 89 = \frac{\log 89}{\log 5} \approx \frac{1.949390007}{0.6989700043} \approx 2.7889$$

or

$$\log_5 89 = \frac{\ln 89}{\ln 5} \approx \frac{4.48863637}{1.609437912} \approx 2.7889$$

(b) $$\log_{\sqrt{2}} \sqrt{5} = \frac{\log \sqrt{5}}{\log \sqrt{2}} = \frac{\frac{1}{2}\log 5}{\frac{1}{2}\log 2} = \frac{\log 5}{\log 2} \approx 2.3219$$

or

$$\log_{\sqrt{2}} \sqrt{5} = \frac{\ln \sqrt{5}}{\ln \sqrt{2}} = \frac{\frac{1}{2}\ln 5}{\frac{1}{2}\ln 2} = \frac{\ln 5}{\ln 2} \approx 2.3219$$

SUMMARY Properties of Logarithms

In the list that follows, a, b, M, N, and r are real numbers. Also, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, $M > 0$, and $N > 0$.

Definition	$y = \log_a x$ means $x = a^y$	
Properties of logarithms	$\log_a 1 = 0; \quad \log_a a = 1$	$\log_a M^r = r \log_a M$
	$a^{\log_a M} = M; \quad \log_a a^r = r$	$a^x = e^{x \ln a}$
	$\log_a(MN) = \log_a M + \log_a N$	If $M = N$, then $\log_a M = \log_a N$.
	$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$	If $\log_a M = \log_a N$, then $M = N$.
Change-of-Base Formula	$\log_a M = \frac{\log_b M}{\log_b a}$	