Section 5.8

Exponential Growth and Decay Models;

Newton's Law;

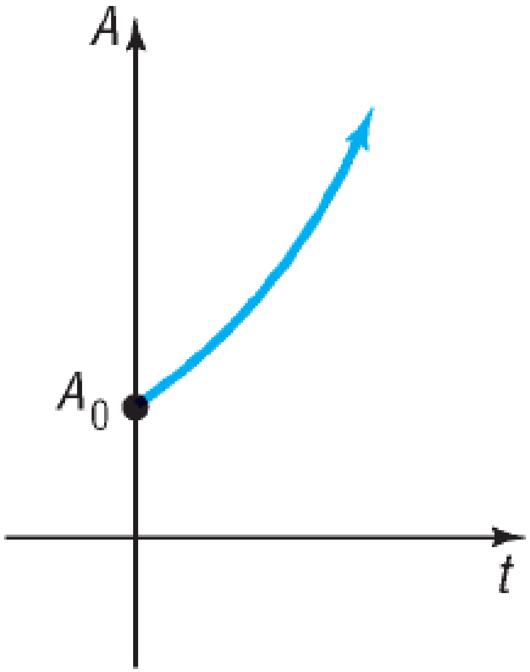
Logistic Growth and Decay Models

PEARSON

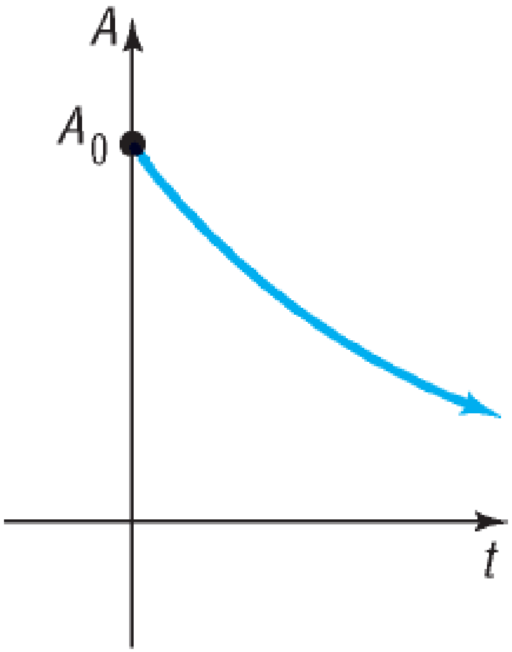
1 Find Equations of Populations That Obey the Law of Uninhibited Growth

Uninhibited Growth

$$A(t) = A_0 e^{kt} \quad \textbf{(1)}$$



(a) $A(t) = A_0 e^{kt}, k > 0$



(b) $A(t) = A_0 e^{kt}, k < 0$

Uninhibited Growth of Cells

A model that gives the number N of cells in a culture after a time t has passed (in the early stages of growth) is

$$N(t) = N_0 e^{kt} \qquad k > 0 \tag{2}$$

where N_0 is the initial number of cells and k is a positive constant that represents the growth rate of the cells.

EXAMPLE **Bacterial Growth**

A colony of bacteria grows according to the law of uninhibited growth according to the function $N(t) = 90e^{0.05t}$, where N is measured in grams and t is measure in days.

(a) Determine the initial amount of bacteria.

(b) What is the growth rate of the bacteria?

(c) What is the population after 5 days?

(d) How long will it take for the population to reach 140 cells?

(e) What is the doubling time for the population?

(a) $N(0) = 90e^{0.05(0)} = 90$ cells

(b) $k = 0.05$ so the growth rate is 5%.

(c) $N(5) = 90e^{0.05(5)} = 116$ cells

(d) $140 = 90e^{0.05t}$ $\quad \dfrac{140}{90} = e^{0.05t}$

$$\ln\left(\frac{14}{9}\right) = 0.05t \qquad t = \frac{\ln\left(\frac{14}{9}\right)}{0.05} \approx 8.8 \text{ days}$$

(e) $180 = 90e^{0.05t}$

$$\frac{180}{90} = e^{0.05t} \qquad \ln 2 = 0.05t$$

$$t = \frac{\ln 2}{0.05} \approx 13.9 \text{ days}$$

EXAMPLE **Bacterial Growth**

A colony of bacteria increases according to the law of uninhibited growth.

(a) If N is the number of cells and t is the time in hours, express N as a function of t.

(b) If the number of bacteria doubles in 2 hours, find the function that gives the number of cells in the culture.

(c) How long will it take for the size of the colony to triple?

(d) How long will it take for the population to double a second time (that is increase four times)?

(a) $N(t) = N_0 e^{kt}$

(b) $2N_0 = N_0 e^{k(2)}$

$2 = e^{2k} \quad \ln 2 = 2k$

$$k = \frac{\ln 2}{2} \approx 0.3465736$$

$$N(t) = N_0 e^{0.3465736t}$$

(c) $3N_0 = N_0 e^{0.3465736t}$

$3 = e^{0.3465736t} \qquad \ln 3 = 0.3465736t$

$$t = \frac{\ln 3}{0.3465736} \approx 3.2 \text{ hrs}$$

(d) If the population doubles in 2 hours, it will double a second time in 2 more hours for a total time of 4 hours.

2 Find Equations of Populations That Obey the Law of Decay

Uninhibited Radioactive Decay

The amount A of a radioactive material present at time t is given by

$$\boxed{A(t) = A_0 e^{kt} \qquad k < 0 \qquad \textbf{(3)}}$$

where A_0 is the original amount of radioactive material and k is a negative number that represents the rate of decay.

EXAMPLE Estimating the Age of Ancient Tools

Traces of burned wood along with ancient stone tools in an archeological dig in Chile were found to contain approximately 1.67% of the original amount of carbon 14. If the half-life of carbon 14 is 5600 years, approximately when was the tree cut and burned?

$$\frac{1}{2}A_0 = A_0 e^{k(5600)}$$

$$\frac{1}{2} = e^{5600k}$$

$$\ln\frac{1}{2} = 5600k$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5600} \approx -0.000124$$

$$\boxed{A(t) = A_0 e^{kt}, \qquad k < 0}$$

$$0.0167 A_0 = A_0 e^{-0.000124t}$$

$$0.0167 = e^{-0.000124t}$$

$$\ln 0.0167 = -0.000124t$$

$$t = \frac{\ln 0.0167}{-0.000124} \approx 33{,}000 \text{ years}$$

3 Use Newton’s Law of Cooling

Newton's Law of Cooling

The temperature u of a heated object at a given time t can be modeled by the following function:

$$u(t) = T + (u_0 - T)e^{kt} \qquad k < 0 \qquad \textbf{(4)}$$

where T is the constant temperature of the surrounding medium, u_0 is the initial temperature of the heated object, and k is a negative constant.

EXAMPLE **Using Newton's Law of Cooling**

A cake is heated to 350° F and is then allowed to cool in a room whose air temperature is 70° F.

(a)If the temperature of the cake is 300° F after 5 minutes, when will its temperature be 200° F?

$$\text{(a)}\quad u(t) = 70 + (350 - 70)e^{kt}$$

$$300 = 70 + (350 - 70)e^{k(5)}$$

Use $u(5) = 300$ to find k

$$300 = 70 + 280e^{5k}$$

$$\frac{230}{280} = e^{5k} \qquad \ln\left(\frac{230}{280}\right) = 5k$$

$$k = \frac{1}{5}\ln\left(\frac{23}{28}\right) \approx -0.039342$$

$$\boxed{u(t) = T + (u_0 - T)e^{kt}, \qquad k < 0}$$

$$u(t) = 70 + 280e^{-0.039342t}$$

$$200 = 70 + 280e^{-0.039342t}$$

$$\frac{130}{280} = e^{-0.039342t}$$

$$\ln\left(\frac{130}{280}\right) = -0.039342t$$

$$t = \frac{\ln\left(\frac{130}{280}\right)}{-0.039342} \approx 19.5 \text{ minutes}$$

EXAMPLE Using Newton's Law of Cooling

A cake is heated to 350° F and is then allowed to cool in a room whose air temperature is 70° F.

(b)Determine the elapsed time before the temperature of the cake is 100° F.

(c)What do you notice about the temperature as time passes?

(b) $u(t) = 70 + 280e^{-0.039342t}$

$$100 = 70 + 280e^{-0.039342t}$$

$$\frac{30}{280} = e^{-0.039342t}$$

$$\ln\left(\frac{30}{280}\right) = -0.039342t$$

$$t = \frac{\ln\left(\frac{30}{280}\right)}{-0.039342} \approx 56.8 \text{ minutes}$$

(c) Looking at the function

$$u(t) = 70 + 280e^{-0.039342t}$$

we see as t increases the exponential $e^{-0.039342t}$ approaches zero so the temperature of the cake approaches the room temperature of 70°F.

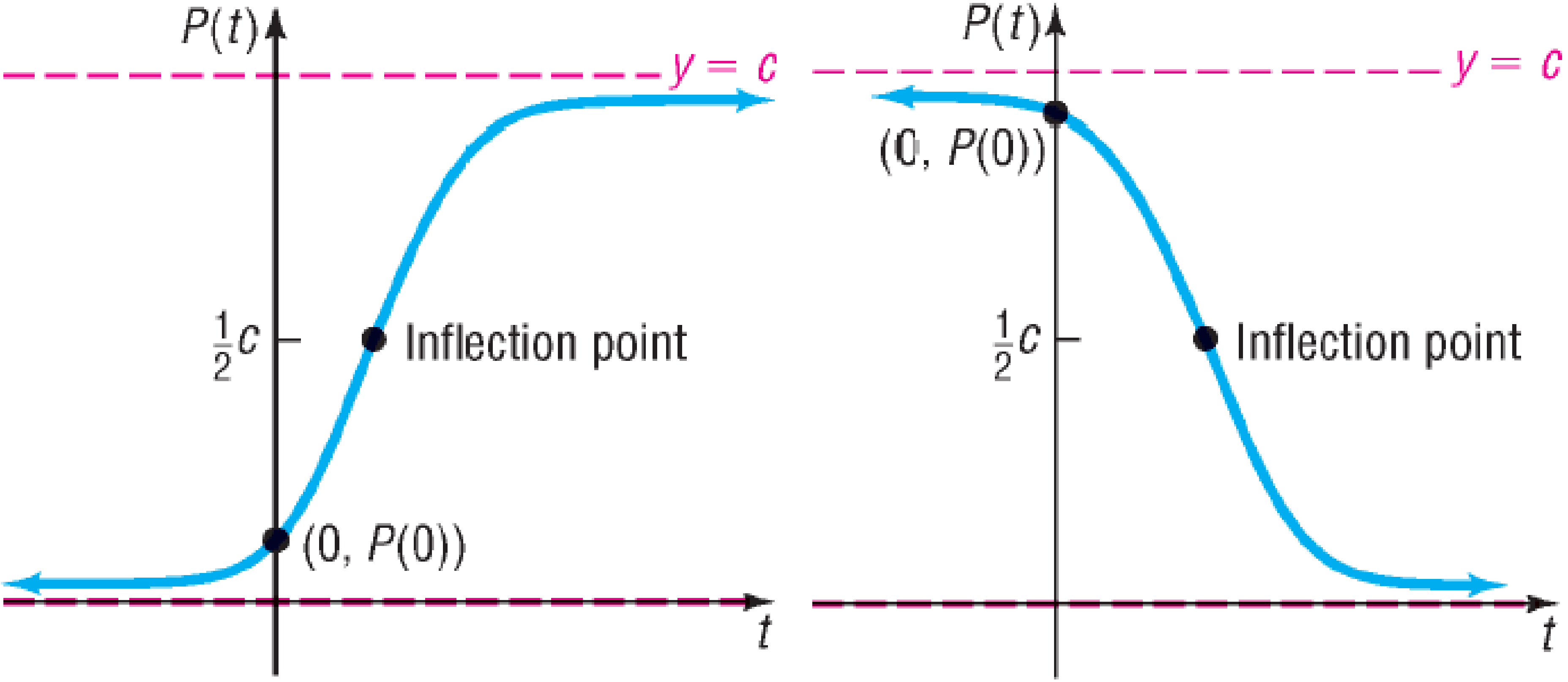
4 Use Logistic Models

Logistic Model

In a logistic model, the population P after time t is given by the function

$$P(t) = \frac{c}{1 + ae^{-bt}} \qquad \textbf{(6)}$$

where a, b, and c are constants with $a > 0$ and $c > 0$. The model is a growth model if $b > 0$; the model is a decay model if $b < 0$.



P(t)
y = c
1/2 c
Inflection point
(0, P(0))
t
P(t)
y = c
(0, P(0))
1/2 c
Inflection point
t

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

Properties of the Logistic Growth Function

1. The domain is the set of all real numbers. The range is the interval $(0, c)$, where c is the carrying capacity.
2. There are no x-intercepts; the y-intercept is $P(0)$.
3. There are two horizontal asymptotes: $y = 0$ and $y = c$.
4. $P(t)$ is an increasing function if $b > 0$ and a decreasing function if $b < 0$.
5. There is an **inflection point** where $P(t)$ equals $\frac{1}{2}$ of the carrying capacity. The inflection point is the point on the graph where the graph changes from being curved upward to curved downward for growth functions and the point where the graph changes from being curved downward to curved upward for decay functions.
6. The graph is smooth and continuous, with no corners or gaps.

EXAMPLE Fruit Fly Population

Fruit flies are placed in a half-pint milk bottle with a banana (for food) and yeast plants (for food and to provide a stimulus to lay eggs). Suppose that the fruit fly population after t days is given by

$$P(t) = \frac{230}{1 + 56.5e^{-0.37t}}$$

(a) State the carrying capacity and the growth rate.
(b) Determine the initial population.
(c) What is the population after 5 days?

(a) As $t \to \infty$, $e^{-0.37t} \to 0$ so $P(t) \to 230$ which is the carrying capacity of this bottle of fruit flies.
The growth rate is $|0.37| = 37\%$

(b) $P(0) = \dfrac{230}{1+56.5e^{-0.37(0)}} = \dfrac{230}{1+56.5} = 4,$ so initally there are 4 fruit flies.

(c) $P(5) = \dfrac{230}{1+56.5e^{-0.37(5)}} \approx$ 23 fruit flies

EXAMPLE Fruit Fly Population

Fruit flies are placed in a half-pint milk bottle with a banana (for food) and yeast plants (for food and to provide a stimulus to lay eggs). Suppose that the fruit fly population after t days is given by

$$P(t) = \frac{230}{1 + 56.5e^{-0.37t}}$$

(d) How long does it take for the population to reach 180?

(e) Use a graphing utility to determine how long it takes for the population to reach one-half of the carrying capacity by graphing $Y_1 = P(t)$ and $Y_2 = 115$ and using INTERSECT.

(d) $$180 = \frac{230}{1+56.5e^{-0.37t}}$$

$$180\left(1+56.5e^{-0.37t}\right) = 230$$

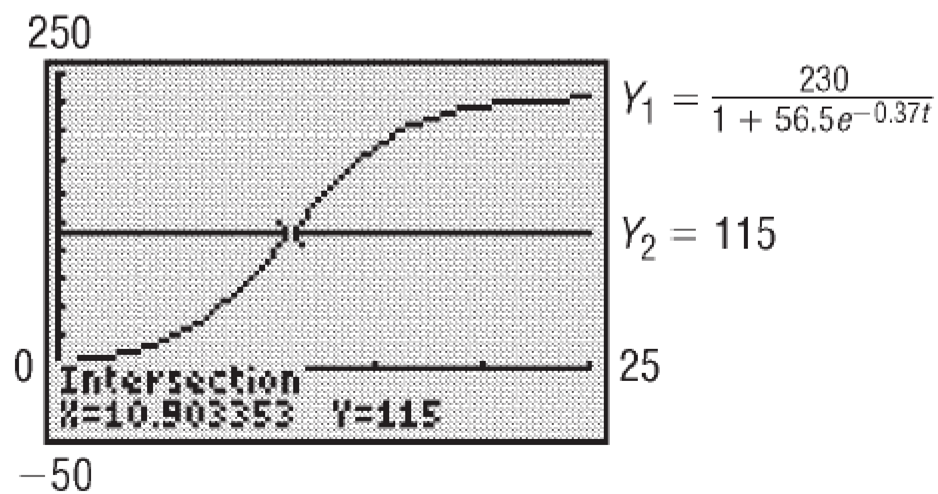
$$1.2778 = 1 + 56.5e^{-0.37t}$$

$$0.2778 = 56.5e^{-0.37t}$$

$$0.0049 = e^{-0.37t}$$

$$\ln 0.0049 = -0.37t$$

$$t = \frac{\ln 0.0049}{-0.37} \approx 14.4 \text{ days}$$

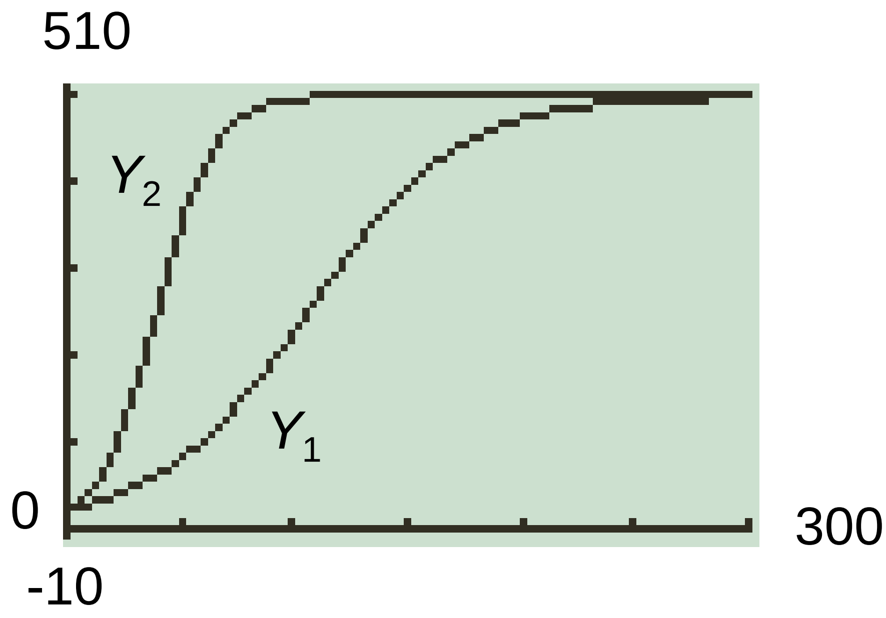


Exploration

On the same viewing rectangle, graph

$$Y_1 = \frac{500}{1 + 24e^{-0.03t}} \quad \text{and} \quad Y_2 = \frac{500}{1 + 24e^{-0.08t}}$$

What effect does the growth rate $|b|$ have on the logistic growth function?



EXAMPLE **Wood Products**

The EFISCEN wood product model classifies wood products according to their life-span. There are four classifications: short (1 year), medium short (4 years), medium long (16 years), and long (50 years). Based on data obtained from the European Forest Institute, the percentage of remaining wood products after t years for wood products with long life-spans (such as those used in the building industry) is given by

$$P(t) = \frac{100.3952}{1 + 0.0316e^{0.0581t}}$$

(a) What is the decay rate?

(b) What is the percentage of remaining wood products after 10 years?

(a) The decay rate is $|-0.0581| = 5.81\%$

(b) $P(10) = \dfrac{100.3952}{1+0.0316e^{0.0581(10)}} = 95.0$ so 95% remain after 10 years.

EXAMPLE Wood Products

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$$P(t) = \frac{100.3952}{1 + 0.0316e^{0.0581t}}$$

(c) How long does it take for the percentage of remaining wood products to reach 50%?

(d) Explain why the numerator given in the model is reasonable.

(c) $$50 = \frac{100.3952}{1+0.0316e^{0.0581t}}$$

$$50\left(1+0.0316e^{0.0581t}\right) = 100.3952$$

$$1+0.0316e^{0.0581t} = 2.0079$$

$$0.0316e^{0.0581t} = 1.0079$$

$$e^{0.0581t} = 31.8956$$

$$0.0581t = \ln 31.8956$$

$$t = 59.6 \text{ years}$$

(d) The numerator of 100.3952 is reasonable because the maximum percentage of wood products remaining that is possible is 100%.