Chapter 7

Analytic Trigonometry

Section 7.1

1. Domain: $\{x|x \text{ is any real number}\}$;

Range: $\{y|-1 \le y \le 1\}$

2. $\{x \mid x \ge 1\}$ or $\{x \mid x \le 1\}$

3. $[3, \infty)$

4. True

5. 1; $\frac{\sqrt{3}}{2}$

6. $-\frac{1}{2}$; -1

7. $x = \sin y$

8. $0 \le x \le \pi$

9. $-\infty \le x \le \infty$

10. False. The domain of $y = \sin^{-1} x$ is $-1 \le x \le 1$.

11. True

12. True

13. d

14. a

15. $\sin^{-1} 0$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals 0.

$\sin\theta = 0, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\theta = 0$

$\sin^{-1} 0 = 0$

16. $\cos^{-1} 1$

We are finding the angle θ, $0 \le \theta \le \pi$, whose cosine equals 1.

$\cos\theta = 1, \quad 0 \le \theta \le \pi$

$\theta = 0$

$\cos^{-1} 1 = 0$

17. $\sin^{-1}(-1)$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals -1.

$\sin\theta = -1, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\theta = -\frac{\pi}{2}$

$\sin^{-1}(-1) = -\frac{\pi}{2}$

18. $\cos^{-1}(-1)$

We are finding the angle θ, $0 \le \theta \le \pi$, whose cosine equals -1.

$\cos\theta = -1, \quad 0 \le \theta \le \pi$

$\theta = \pi$

$\cos^{-1}(-1) = \pi$

19. $\tan^{-1} 0$

We are finding the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 0.

$\tan\theta = 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\theta = 0$

$\tan^{-1} 0 = 0$

20. $\tan^{-1}(-1)$

We are finding the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals -1.

$\tan\theta = -1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\theta = -\frac{\pi}{4}$

$\tan^{-1}(-1) = -\frac{\pi}{4}$

21. $\sin^{-1}\frac{\sqrt{2}}{2}$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{\sqrt{2}}{2}$.

$$\sin\theta = \frac{\sqrt{2}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = \frac{\pi}{4}$$
$$\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

22. $\tan^{-1}\frac{\sqrt{3}}{3}$

We are finding the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $\frac{\sqrt{3}}{3}$.

$$\tan\theta = \frac{\sqrt{3}}{3}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = \frac{\pi}{6}$$
$$\tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$$

23. $\tan^{-1}\sqrt{3}$

We are finding the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $\sqrt{3}$.

$$\tan\theta = \sqrt{3}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = \frac{\pi}{3}$$
$$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

24. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta = -\frac{\sqrt{3}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{3}$$
$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

25. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

We are finding the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

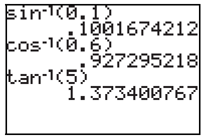
$$\cos\theta = -\frac{\sqrt{3}}{2}, \quad 0 \le \theta \le \pi$$
$$\theta = \frac{5\pi}{6}$$
$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

26. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{2}}{2}$.

$$\sin\theta = -\frac{\sqrt{2}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{4}$$
$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

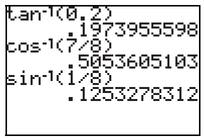
27. $\sin^{-1} 0.1 \approx 0.10$



28. $\cos^{-1} 0.6 \approx 0.93$

29. $\tan^{-1} 5 \approx 1.37$

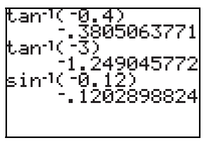
30. $\tan^{-1} 0.2 \approx 0.20$



31. $\cos^{-1} \frac{7}{8} \approx 0.51$

32. $\sin^{-1} \frac{1}{8} \approx 0.13$

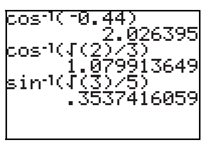
33. $\tan^{-1}(-0.4) \approx -0.38$



34. $\tan^{-1}(-3) \approx -1.25$

35. $\sin^{-1}(-0.12) \approx -0.12$

36. $\cos^{-1}(-0.44) \approx 2.03$



37. $\cos^{-1} \frac{\sqrt{2}}{3} \approx 1.08$

38. $\sin^{-1} \frac{\sqrt{3}}{5} \approx 0.35$

39. $\cos^{-1}\left(\cos \frac{4\pi}{5}\right)$ follows the form of the equation $f^{-1}(f(x)) = \cos^{-1}(\cos(x)) = x$. Since $\frac{4\pi}{5}$ is in the interval $[0, \pi]$, we can apply the equation directly and get $\cos^{-1}\left(\cos \frac{4\pi}{5}\right) = \frac{4\pi}{5}$.

40. $\sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right)$ follows the form of the equation $f^{-1}(f(x)) = \sin^{-1}(\sin(x)) = x$. Since $-\frac{\pi}{10}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can apply the equation directly and get
$\sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right) = -\frac{\pi}{10}$.

41. $\tan^{-1}\left(\tan\left(-\frac{3\pi}{8}\right)\right)$ follows the form of the equation $f^{-1}(f(x)) = \tan^{-1}(\tan(x)) = x$. Since $-\frac{3\pi}{8}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can apply the equation directly and get
$\tan^{-1}\left(\tan\left(-\frac{3\pi}{8}\right)\right) = -\frac{3\pi}{8}$.

42. $\sin^{-1}\left(\sin\left(-\frac{3\pi}{7}\right)\right)$ follows the form of the equation $f^{-1}(f(x)) = \sin^{-1}(\sin(x)) = x$. Since $-\frac{3\pi}{7}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can apply the equation directly and get
$\sin^{-1}\left(\sin\left(-\frac{3\pi}{7}\right)\right) = -\frac{3\pi}{7}$.

43. $\sin^{-1}\left(\sin\left(\frac{9\pi}{8}\right)\right)$ follows the form of the equation $f^{-1}(f(x)) = \sin^{-1}(\sin(x)) = x$, but we cannot use the formula directly since $\frac{9\pi}{8}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin \frac{9\pi}{8} = \sin\theta$. The angle $\frac{9\pi}{8}$ is in quadrant III so sine is negative. The reference angle of $\frac{9\pi}{8}$ is $\frac{\pi}{8}$ and we want θ to be in quadrant IV so sine will still be negative. Thus, we have
$\sin \frac{9\pi}{8} = \sin\left(-\frac{\pi}{8}\right)$. Since $-\frac{\pi}{8}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can apply the equation above and get $\sin^{-1}\left(\sin \frac{9\pi}{8}\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{8}\right)\right) = -\frac{\pi}{8}$.

44. $\sin^{-1}\left(\sin\left(\frac{11\pi}{4}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\sin^{-1}\left(\sin\left(x\right)\right)=x$, but we cannot use the formula directly since $\frac{11\pi}{4}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\sin\frac{11\pi}{4}=\sin\theta$. The angle $\frac{11\pi}{4}$ is in quadrant II so sine is positive. The reference angle of $\frac{11\pi}{4}$ is $\frac{3\pi}{4}$ and we need θ to be in quadrant I so sine will still be positive. Thus, we have $\sin\frac{3\pi}{4}=\sin\left(\frac{\pi}{4}\right)$. Since $\frac{\pi}{4}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get $\sin^{-1}\left(\sin\frac{11\pi}{4}\right)=\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)=\frac{\pi}{4}$.

45. $\cos^{-1}\left(\cos\left(-\frac{5\pi}{3}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\cos^{-1}\left(\cos\left(x\right)\right)=x$, but we cannot use the formula directly since $-\frac{5\pi}{3}$ is not in the interval $\left[0,\pi\right]$. We need to find an angle θ in the interval $\left[0,\pi\right]$ for which $\cos\left(-\frac{5\pi}{3}\right)=\cos\theta$. The angle $-\frac{5\pi}{3}$ is in quadrant I so the reference angle of $-\frac{5\pi}{3}$ is $\frac{\pi}{3}$. Thus, we have $\cos\left(-\frac{5\pi}{3}\right)=\cos\frac{\pi}{3}$. Since $\frac{\pi}{3}$ is in the interval $\left[0,\pi\right]$, we can apply the equation above and get $\cos^{-1}\left(\cos\left(-\frac{5\pi}{3}\right)\right)=\cos^{-1}\left(\cos\frac{\pi}{3}\right)=\frac{\pi}{3}$.

46. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\cos^{-1}\left(\cos\left(x\right)\right)=x$, but we cannot use the formula directly since $\frac{7\pi}{6}$ is not in the interval $\left[0,\pi\right]$. We need to find an angle θ in the interval $\left[0,\pi\right]$ for which $\cos\left(\frac{7\pi}{6}\right)=\cos\theta$. The angle $\frac{7\pi}{6}$ is in quadrant III so the we need an angle in the desired interval whose cosine is equal to the cosine of $\frac{7\pi}{6}$. Thus, we have $\cos\left(\frac{7\pi}{6}\right)=\cos\frac{5\pi}{6}$. Since $\frac{5\pi}{6}$ is in the interval $\left[0,\pi\right]$, we can apply the equation above and get $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)=\cos^{-1}\left(\cos\frac{5\pi}{6}\right)=\frac{5\pi}{6}$.

47. $\tan^{-1}\left(\tan\left(\frac{4\pi}{5}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\tan^{-1}\left(\tan\left(x\right)\right)=x$, but we cannot use the formula directly since $\frac{4\pi}{5}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\tan\left(\frac{4\pi}{5}\right)=\tan\theta$. The angle $\frac{4\pi}{5}$ is in quadrant II so tangent is negative. The reference angle of $\frac{4\pi}{5}$ is $\frac{\pi}{5}$ and we want θ to be in quadrant IV so tangent will still be negative. Thus, we have $\tan\left(\frac{4\pi}{5}\right)=\tan\left(-\frac{\pi}{5}\right)$. Since $-\frac{\pi}{5}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get

$\tan^{-1}\left(\tan\left(\frac{4\pi}{5}\right)\right)=\tan^{-1}\left(\tan\left(-\frac{\pi}{5}\right)\right)=-\frac{\pi}{5}$.

48. $\tan^{-1}\left(\tan\left(-\frac{10\pi}{9}\right)\right)$ follows the form of the equation $f^{-1}(f(x))=\tan^{-1}(\tan(x))=x$, but we cannot use the formula directly since $-\frac{10\pi}{9}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\tan\left(-\frac{10\pi}{9}\right)=\tan\theta$. The angle $-\frac{10\pi}{9}$ is in quadrant II so tangent is negative. The reference angle of $-\frac{10\pi}{9}$ is $\frac{\pi}{9}$ and we want θ to be in quadrant IV so tangent will still be negative. Thus, we have $\tan\left(-\frac{10\pi}{9}\right)=\tan\left(-\frac{\pi}{9}\right)$. Since $-\frac{\pi}{9}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get

$\tan^{-1}\left(\tan\left(-\frac{10\pi}{9}\right)\right)=\tan^{-1}\left(\tan\left(-\frac{\pi}{9}\right)\right)=-\frac{\pi}{9}$.

49. $\tan^{-1}\left(\tan\left(-\frac{2\pi}{3}\right)\right)$ follows the form of the equation $f^{-1}(f(x))=\tan^{-1}(\tan(x))=x$. but we cannot use the formula directly since $-\frac{2\pi}{3}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\tan\left(-\frac{2\pi}{3}\right)=\tan\theta$. The angle $-\frac{2\pi}{3}$ is in quadrant III so tangent is positive. The reference angle of $-\frac{2\pi}{3}$ is $\frac{\pi}{3}$ and we want θ to be in quadrant I so tangent will still be positive. Thus, we have $\tan\left(-\frac{2\pi}{3}\right)=\tan\left(\frac{\pi}{3}\right)$. Since $\frac{\pi}{3}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get $\tan^{-1}\left(\tan\left(-\frac{2\pi}{3}\right)\right)=\tan^{-1}\left(\tan\frac{\pi}{3}\right)=\frac{\pi}{3}$.

50. $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$ follows the form of the equation $f^{-1}(f(x))=\cos^{-1}(\cos(x))=x$, but we cannot use the formula directly since $\frac{4\pi}{3}$ is not in the interval $[0,\pi]$. We need to find an angle θ in the interval $[0,\pi]$ for which $\cos\left(\frac{4\pi}{3}\right)=\cos\theta$. The angle $\frac{4\pi}{3}$ is in quadrant III so the reference angle of $\frac{4\pi}{3}$ is $\frac{\pi}{3}$. We want the angle to be in quadrant II and the cosine to be negative. Thus, we have $\cos\left(\frac{4\pi}{3}\right)=\cos\frac{2\pi}{3}$. Since $\frac{2\pi}{3}$ is in the interval $[0,\pi]$, we can apply the equation above and get

$\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)=\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}$.

51. $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$ follows the form of the equation $f^{-1}(f(x))=\cos^{-1}(\cos(x))=x$, but we cannot use the formula directly since $-\frac{\pi}{4}$ is not in the interval $[0,\pi]$. We need to find an angle θ in the interval $[0,\pi]$ for which $\cos\left(-\frac{\pi}{4}\right)=\cos\theta$. The angle $-\frac{\pi}{4}$ is in quadrant IV so the reference angle of $-\frac{\pi}{4}$ is $\frac{\pi}{4}$. Thus, we have $\cos\left(-\frac{\pi}{4}\right)=\cos\frac{\pi}{4}$. Since $\frac{\pi}{4}$ is

in the interval $\left[0,\pi\right]$, we can apply the equation above and get
$\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)=\cos^{-1}\left(\cos\frac{\pi}{4}\right)=\frac{\pi}{4}$.

52. $\sin^{-1}\left(\sin\left(-\frac{3\pi}{4}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\sin^{-1}\left(\sin\left(x\right)\right)=x$, but we cannot use the formula directly since $-\frac{3\pi}{4}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\sin\left(-\frac{3\pi}{4}\right)=\sin\theta$. The reference angle of $-\frac{3\pi}{4}$ is $\frac{\pi}{4}$ and we want θ to be in quadrant IV so sine will still be negative. Thus, we have $\sin\left(-\frac{3\pi}{4}\right)=\sin\left(-\frac{\pi}{4}\right)$. Since $\left(-\frac{\pi}{4}\right)$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get
$\sin^{-1}\left(\sin\left(-\frac{3\pi}{4}\right)\right)=\sin^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right)=-\frac{\pi}{4}$.

53. $\tan^{-1}\left(\tan\left(\frac{\pi}{2}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\tan^{-1}\left(\tan\left(x\right)\right)=x$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\tan\left(\frac{\pi}{2}\right)=\tan\theta$. In this case, $\tan\left(\frac{\pi}{2}\right)$ is undefined so $\tan^{-1}\left(\tan\left(\frac{\pi}{2}\right)\right)$ would also be undefined.

54. $\tan^{-1}\left(\tan\left(-\frac{3\pi}{2}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\tan^{-1}\left(\tan\left(x\right)\right)=x$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\tan\left(-\frac{3\pi}{2}\right)=\tan\theta$. The reference angle of $-\frac{3\pi}{2}$ is $\frac{\pi}{2}$. Thus, we have $\tan\left(-\frac{3\pi}{2}\right)=\tan\left(\frac{\pi}{2}\right)$. In this case, $\tan\left(\frac{\pi}{2}\right)$ is undefined so $\tan^{-1}\left(\tan\left(\frac{\pi}{2}\right)\right)$ would also be undefined.

55. $\sin\left(\sin^{-1}\frac{1}{4}\right)$ follows the form of the equation $f\left(f^{-1}\left(x\right)\right)=\sin\left(\sin^{-1}\left(x\right)\right)=x$. Since $\frac{1}{4}$ is in the interval $\left[-1,1\right]$, we can apply the equation directly and get $\sin\left(\sin^{-1}\frac{1}{4}\right)=\frac{1}{4}$.

56. $\cos\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$ follows the form of the equation $f\left(f^{-1}\left(x\right)\right)=\cos\left(\cos^{-1}\left(x\right)\right)=x$. Since $-\frac{2}{3}$ is in the interval $\left[-1,1\right]$, we can apply the equation directly and get
$\cos\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)=-\frac{2}{3}$.

57. $\tan\left(\tan^{-1}4\right)$ follows the form of the equation $f\left(f^{-1}\left(x\right)\right)=\tan\left(\tan^{-1}\left(x\right)\right)=x$. Since 4 is a real number, we can apply the equation directly and get $\tan\left(\tan^{-1}4\right)=4$.

58. $\tan\left(\tan^{-1}(-2)\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\tan\left(\tan^{-1}(x)\right)=x$. Since -2 is a real number, we can apply the equation directly and get $\tan\left(\tan^{-1}(-2)\right)=-2$.

59. Since there is no angle θ such that $\cos\theta=1.2$, the quantity $\cos^{-1}1.2$ is not defined. Thus, $\cos\left(\cos^{-1}1.2\right)$ is not defined.

60. Since there is no angle θ such that $\sin\theta=-2$, the quantity $\sin^{-1}(-2)$ is not defined. Thus, $\sin\left(\sin^{-1}(-2)\right)$ is not defined.

61. $\tan\left(\tan^{-1}\pi\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\tan\left(\tan^{-1}(x)\right)=x$. Since π is a real number, we can apply the equation directly and get $\tan\left(\tan^{-1}\pi\right)=\pi$.

62. Since there is no angle θ such that $\sin\theta=-1.5$, the quantity $\sin^{-1}(-1.5)$ is not defined. Thus, $\sin\left(\sin^{-1}(-1.5)\right)$ is not defined.

63. $f(x)=5\sin x+2$

$$y=5\sin x+2$$
$$x=5\sin y+2$$
$$5\sin y=x-2$$
$$\sin y=\frac{x-2}{5}$$
$$y=\sin^{-1}\frac{x-2}{5}=f^{-1}(x)$$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ or $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse sine function is $\frac{x-2}{5}$ and that it must lie in the interval $[-1,1]$. That is,

$$-1\le\frac{x-2}{5}\le 1$$
$$-5\le x-2\le 5$$
$$-3\le x\le 7$$

The domain of $f^{-1}(x)$ is $\{x\,|\,-3\le x\le 7\}$, or $[-3,7]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is also $[-3,7]$.

64. $f(x)=2\tan x-3$

$$y=2\tan x-3$$
$$x=2\tan y-3$$
$$2\tan y=x+3$$
$$\tan y=\frac{x+3}{2}$$
$$y=\tan^{-1}\frac{x+3}{2}=f^{-1}(x)$$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\frac{\pi}{2}<x<\frac{\pi}{2}$ or $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse tangent function can be any real number. Thus, the domain of $f^{-1}(x)$ is all real numbers, or $(-\infty,\infty)$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $(-\infty,\infty)$.

65. $f(x)=-2\cos(3x)$

$$y=-2\cos(3x)$$
$$x=-2\cos(3y)$$
$$\cos(3y)=-\frac{x}{2}$$
$$3y=\cos^{-1}\left(-\frac{x}{2}\right)$$
$$y=\frac{1}{3}\cos^{-1}\left(-\frac{x}{2}\right)=f^{-1}(x)$$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $0\le x\le\frac{\pi}{3}$, or $\left[0,\frac{\pi}{3}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse cosine function is $\frac{-x}{2}$ and that it must lie in the interval $[-1,1]$. That is,

$$-1 \le -\frac{x}{2} \le 1$$
$$2 \ge x \ge -2$$
$$-2 \le x \le 2$$

The domain of $f^{-1}(x)$ is $\{x \mid -2 \le x \le 2\}$, or $[-2,2]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $[-2,2]$.

66. $f(x) = 3\sin(2x)$

$$y = 3\sin(2x)$$
$$x = 3\sin(2y)$$
$$\sin(2y) = \frac{x}{3}$$
$$2y = \sin^{-1}\frac{x}{3}$$
$$y = \frac{1}{2}\sin^{-1}\frac{x}{3} = f^{-1}(x)$$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$, or $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse sine function is $\frac{x}{3}$ and that it must lie in the interval $[-1,1]$. That is,

$$-1 \le \frac{x}{3} \le 1$$
$$-3 \le x \le 3$$

The domain of $f^{-1}(x)$ is $\{x \mid -3 \le x \le 3\}$, or $[-3,3]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $[-3,3]$.

67. $f(x) = -\tan(x+1) - 3$

$$y = -\tan(x+1) - 3$$
$$x = -\tan(y+1) - 3$$
$$\tan(y+1) = -x - 3$$
$$y + 1 = \tan^{-1}(-x-3)$$
$$y = -1 + \tan^{-1}(-x-3)$$
$$= -1 - \tan^{-1}(x+3) = f^{-1}(x)$$

(note here we used the fact that $y = \tan^{-1} x$ is an odd function).

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-1-\frac{\pi}{2} \le x \le \frac{\pi}{2} - 1$, or $\left[-1-\frac{\pi}{2}, \frac{\pi}{2} - 1\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse tangent function can be any real number. Thus, the domain of $f^{-1}(x)$ is all real numbers, or $(-\infty, \infty)$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $(-\infty, \infty)$.

68. $f(x) = \cos(x+2) + 1$

$$y = \cos(x+2) + 1$$
$$x = \cos(y+2) + 1$$
$$\cos(y+2) = x - 1$$
$$y + 2 = \cos^{-1}(x-1)$$
$$y = \cos^{-1}(x-1) - 2$$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-2 \le x \le \pi - 2$, or $[-2, \pi-2]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse cosine function is $x-1$ and that it must lie in the interval $[-1,1]$. That is, $-1 \le x - 1 \le 1$

$$0 \le x \le 2$$

The domain of $f^{-1}(x)$ is $\{x \mid 0 \le x \le 2\}$, or $[0,2]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $\left[0,2\right]$.

69. $f\left(x\right)=3\sin\left(2x+1\right)$

$y=3\sin\left(2x+1\right)$

$x=3\sin\left(2y+1\right)$

$\sin\left(2y+1\right)=\frac{x}{3}$

$2y+1=\sin^{-1}\frac{x}{3}$

$2y=\sin^{-1}\left(\frac{x}{3}\right)-1$

$y=\frac{1}{2}\sin^{-1}\left(\frac{x}{3}\right)-\frac{1}{2}=f^{-1}\left(x\right)$

The domain of $f\left(x\right)$ equals the range of $f^{-1}(x)$ and is $-\frac{1}{2}-\frac{\pi}{4}\le x\le-\frac{1}{2}+\frac{\pi}{4}$, or $\left[-\frac{1}{2}-\frac{\pi}{4},-\frac{1}{2}+\frac{\pi}{4}\right]$ in interval notation. To find the domain of $f^{-1}\left(x\right)$ we note that the argument of the inverse sine function is $\frac{x}{3}$ and that it must lie in the interval $\left[-1,1\right]$. That is,

$-1\le\frac{x}{3}\le1$

$-3\le x\le3$

The domain of $f^{-1}\left(x\right)$ is $\left\{x\,|-3\le x\le3\right\}$, or $\left[-3,3\right]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $\left[-3,3\right]$.

70. $f\left(x\right)=2\cos\left(3x+2\right)$

$y=2\cos\left(3x+2\right)$

$x=2\cos\left(3y+2\right)$

$\cos\left(3y+2\right)=\frac{x}{2}$

$3y+2=\cos^{-1}\left(\frac{x}{2}\right)$

$3y=\cos^{-1}\left(\frac{x}{2}\right)-2$

$y=\frac{1}{3}\cos^{-1}\left(\frac{x}{2}\right)-\frac{2}{3}=f^{-1}\left(x\right)$

The domain of $f\left(x\right)$ equals the range of $f^{-1}(x)$ and is $-\frac{2}{3}\le x\le-\frac{2}{3}+\frac{\pi}{3}$, or $\left[-\frac{2}{3},-\frac{2}{3}+\frac{\pi}{3}\right]$ in interval notation. To find the domain of $f^{-1}\left(x\right)$ we note that the argument of the inverse cosine function is $\frac{x}{2}$ and that it must lie in the interval $\left[-1,1\right]$. That is,

$-1\le\frac{x}{2}\le1$

$-2\le x\le2$

The domain of $f^{-1}\left(x\right)$ is $\left\{x\,|-2\le x\le2\right\}$, or $\left[-2,2\right]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $\left[-2,2\right]$.

71. $4\sin^{-1}x=\pi$

$\sin^{-1}x=\frac{\pi}{4}$

$x=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

The solution set is $\left\{\frac{\sqrt{2}}{2}\right\}$.

72. $2\cos^{-1} x = \pi$

$\cos^{-1} x = \frac{\pi}{2}$

$x = \cos\frac{\pi}{2} = 0$

The solution set is $\{0\}$.

73. $3\cos^{-1}(2x) = 2\pi$

$\cos^{-1}(2x) = \frac{2\pi}{3}$

$2x = \cos\frac{2\pi}{3}$

$2x = -\frac{1}{2}$

$x = -\frac{1}{4}$

The solution set is $\left\{-\frac{1}{4}\right\}$.

74. $-6\sin^{-1}(3x) = \pi$

$\sin^{-1}(3x) = -\frac{\pi}{6}$

$3x = \sin\left(-\frac{\pi}{6}\right)$

$3x = -\frac{1}{2}$

$x = -\frac{1}{6}$

The solution set is $\left\{-\frac{1}{6}\right\}$.

75. $3\tan^{-1} x = \pi$

$\tan^{-1} x = \frac{\pi}{3}$

$x = \tan\frac{\pi}{3} = \sqrt{3}$

The solution set is $\left\{\sqrt{3}\right\}$.

76. $-4\tan^{-1} x = \pi$

$\tan^{-1} x = -\frac{\pi}{4}$

$x = \tan\left(-\frac{\pi}{4}\right) = -1$

The solution set is $\{-1\}$.

77. $4\cos^{-1} x - 2\pi = 2\cos^{-1} x$

$2\cos^{-1} x - 2\pi = 0$

$2\cos^{-1} x = 2\pi$

$\cos^{-1} x = \pi$

$x = \cos\pi = -1$

The solution set is $\{-1\}$.

78. $5\sin^{-1} x - 2\pi = 2\sin^{-1} x - 3\pi$

$3\sin^{-1} x = -\pi$

$\sin^{-1} x = -\frac{\pi}{3}$

$x = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

The solution set is $\left\{-\frac{\sqrt{3}}{2}\right\}$.

79. Note that $\theta = 29°45' = 29.75°$.

a. $D = 24 \cdot \left[1 - \frac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(29.75 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right]$

≈ 13.92 hours or 13 hours, 55 minutes

b. $D = 24 \cdot \left[1 - \frac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(29.75 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right]$

≈ 12 hours

c. $D = 24 \cdot \left[1 - \frac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(29.75 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right]$

≈ 13.85 hours or 13 hours, 51 minutes

80. Note that $\theta = 40°45' = 40.75°$.

a. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(40.75 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 14.93 hours or 14 hours, 56 minutes

b. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(40.75 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

c. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(40.75 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 14.83 hours or 14 hours, 50 minutes

81. Note that $\theta = 21°18' = 21.3°$.

a. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(21.3 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 13.30 hours or 13 hours, 18 minutes

b. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(21.3 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

c. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(21.3 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 13.26 hours or 13 hours, 15 minutes

82. Note that $\theta = 61°10' \approx 61.167°$.

a. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(61.167 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 18.96 hours or 18 hours, 57 minutes

b. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(61.167 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

c. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(61.167 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 18.64 hours or 18 hours, 38 minutes

83. a. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(0 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

b. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(0 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

c. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(0 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

d. There are approximately 12 hours of daylight every day at the equator.

84. Note that $\theta = 66°30' = 66.5°$.

a. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(66.5 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 24 hours

b. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(66.5 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

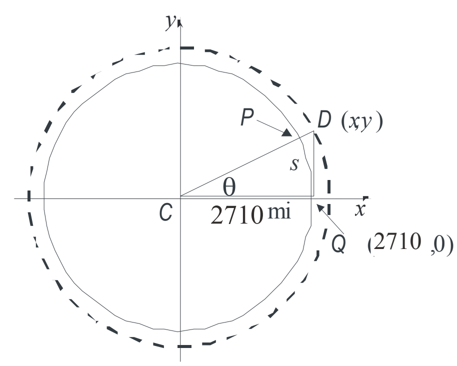
≈ 12 hours

c. $D = 24 \cdot \left(1 - \frac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(66.5 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 22.02 hours or 22 hours, 1 minute

d. The amount of daylight at this location on the winter solstice is $24 - 24 = 0$ hours. That is, on the winter solstice, there is no daylight. In general, for a location at $66°30'$ north latitude, it ranges from around-the-clock daylight to no daylight at all.

85. Let point *C* represent the point on the Earth's axis at the same latitude as Cadillac Mountain, and arrange the figure so that segment *CQ* lies along the *x*-axis (see figure).



At the latitude of Cadillac Mountain, the effective radius of the Earth is 2710 miles. If point $D(x, y)$ represents the peak of Cadillac Mountain, then the length of segment *PD* is

$1530 \text{ ft} \cdot \frac{1 \text{ mile}}{5280 \text{ feet}} \approx 0.29 \text{ mile}$. Therefore, the point $D(x, y) = (2710, y)$ lies on a circle with radius $r = 2710.29$ miles. We now have

$$\cos\theta = \frac{x}{r} = \frac{2710}{2710.29}$$

$$\theta = \cos^{-1}\left(\frac{2710}{2710.29}\right) \approx 0.01463 \text{ radians}$$

Finally, $s = r\theta = 2710(0.01463) \approx 39.64$ miles,

and $\frac{2\pi(2710)}{24} = \frac{39.64}{t}$, so

$$t = \frac{24(39.64)}{2\pi(2710)} \approx 0.05587 \text{ hours} \approx 3.35 \text{ minutes}$$

Therefore, a person atop Cadillac Mountain will see the first rays of sunlight about 3.35 minutes sooner than a person standing below at sea level.

86. $\theta(x) = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right)$.

a. $\theta(10) = \tan^{-1}\left(\frac{34}{10}\right) - \tan^{-1}\left(\frac{6}{10}\right) \approx 42.6°$

If you sit 10 feet from the screen, then the viewing angle is about $42.6°$.

$\theta(15) = \tan^{-1}\left(\frac{34}{15}\right) - \tan^{-1}\left(\frac{6}{15}\right) \approx 44.4°$

If you sit 15 feet from the screen, then the viewing angle is about $44.4°$.

$\theta(20) = \tan^{-1}\left(\frac{34}{20}\right) - \tan^{-1}\left(\frac{6}{20}\right) \approx 42.8°$

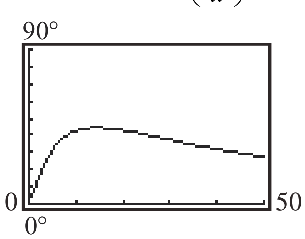
If you sit 20 feet from the screen, then the viewing angle is about $42.8°$.

b. Let r = the row that result in the largest viewing angle. Looking ahead to part (c), we see that the maximum viewing angle occurs when the distance from the screen is about 14.3 feet. Thus,

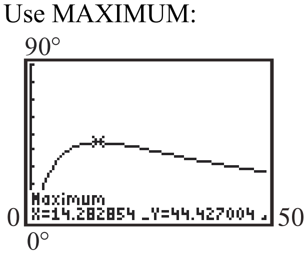
$$5 + 3(r-1) = 14.3$$
$$5 + 3r - 3 = 14.3$$
$$3r = 12.3$$
$$r = 4.1$$

Sitting in the 4th row should provide the largest viewing angle.

c. Set the graphing calculator in degree mode and let $Y_1 = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right)$:



Use MAXIMUM:



The maximum viewing angle will occur when $x \approx 14.3$ feet.

87. a. $a = 0$; $b = \sqrt{3}$; The area is:

$$\tan^{-1} b - \tan^{-1} a = \tan^{-1}\sqrt{3} - \tan^{-1} 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3} \text{ square units}$$

b. $a = -\frac{\sqrt{3}}{3}$; $b = 1$; The area is:

$$\tan^{-1} b - \tan^{-1} a = \tan^{-1} 1 - \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = \frac{\pi}{4} - \left(-\frac{\pi}{6}\right) = \frac{5\pi}{12} \text{ square units}$$

88. a. $a = 0$; $b = \frac{\sqrt{3}}{2}$; The area is:

$$\sin^{-1} b - \sin^{-1} a = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1} 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3} \text{ square units}$$

b. $a = -\frac{1}{2}$; $b = \frac{1}{2}$; The area is:

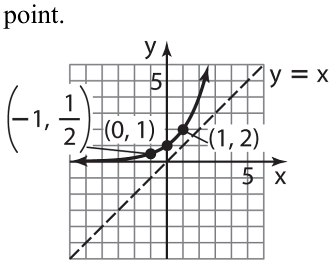
$$\sin^{-1} b - \sin^{-1} a = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3} \text{ square units}$$

89. Here we have $\alpha_1 = 41°50'$, $\beta_1 = -87°37'$, $\alpha_2 = 21°18'$, and $\beta_2 = -157°50'$. Converting minutes to degrees gives $\alpha_1 = \left(41\tfrac{5}{6}\right)°$, $\beta_1 = \left(-87\tfrac{37}{60}\right)°$, $\alpha_2 = 21.3°$, and $\beta_2 = \left(-157\tfrac{5}{6}\right)°$. Substituting these values, and $r = 3960$, into our equation gives $d \approx 4250$ miles. The distance from Chicago to Honolulu is about 4250 miles.
(remember that S and W angles are negative)

90. Here we have $\alpha_1 = 21°18'$, $\beta_1 = -157°50'$, $\alpha_2 = -37°47'$, and $\beta_2 = 144°58'$.
Converting minutes to degrees gives $\alpha_1 = 21.3°$, $\beta_1 = \left(-157\frac{5}{6}\right)°$, $\alpha_2 = \left(-37\frac{47}{60}\right)°$, and $\beta_2 = \left(144\frac{29}{30}\right)°$. Substituting these values, and $r = 3960$, into our equation gives $d \approx 5518$ miles. The distance from Honolulu to Melbourne is about 5518 miles.
(remember that S and W angles are negative)

91.
$$10^{3x} + 4 = 11$$
$$10^{3x} = 7$$
$$\log 10^{3x} = \log 7$$
$$3x \log 10 = \log 7$$
$$3x = \log 7$$
$$x = \frac{\log 7}{3}$$
The solution is: $\left\{\frac{\log 7}{3}\right\}$

92. The function *f* is one-to-one because every horizontal line intersects the graph at exactly one point.



93.
$$f(x) = 1 + 2^x$$
$$y = 1 + 2^x$$
$$x = 1 + 2^y$$
$$x - 1 = 2^y$$
$$\log_2(x-1) = \log_2 2^y$$
$$\log_2(x-1) = y\log_2 2$$
$$\log_2(x-1) = y$$
$$f^{-1}(x) = \log_2(x-1)$$

94. $\sin\frac{\pi}{3}\cos\frac{\pi}{3} = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$

Section 7.2

1. Domain: $\left\{x \,\middle|\, x \neq \text{odd integer multiples of } \frac{\pi}{2}\right\}$,
   Range: $\{y \mid y \le -1 \text{ or } y \ge 1\}$

2. True

3. $\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

4. $x = \sec y$, ≥ 1, 0, π

5. cosine

6. False

7. True

8. True

9. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

   Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{\sqrt{2}}{2}$.
$$\sin\theta = \frac{\sqrt{2}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = \frac{\pi}{4}$$
$$\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

10. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

    Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $\frac{1}{2}$.

$$\cos\theta = \frac{1}{2}, \quad 0 \le \theta \le \pi$$

$$\theta = \frac{\pi}{3}$$

$$\sin\left(\cos^{-1}\frac{1}{2}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

11. $\tan\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

$$\cos\theta = -\frac{\sqrt{3}}{2}, \quad 0 \le \theta \le \pi$$

$$\theta = \frac{5\pi}{6}$$

$$\tan\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$$

12. $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{6}$$

$$\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

13. $\sec\left(\cos^{-1}\frac{1}{2}\right)$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $\frac{1}{2}$.

$$\cos\theta = \frac{1}{2}, \quad 0 \le \theta \le \pi$$

$$\theta = \frac{\pi}{3}$$

$$\sec\left(\cos^{-1}\frac{1}{2}\right) = \sec\frac{\pi}{3} = 2$$

14. $\cot\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{6}$$

$$\cot\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$$

15. $\csc\left(\tan^{-1} 1\right)$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 1.

$$\tan\theta = 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

$$\csc\left(\tan^{-1} 1\right) = \csc\frac{\pi}{4} = \sqrt{2}$$

16. $\sec\left(\tan^{-1}\sqrt{3}\right)$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $\sqrt{3}$.

$$\tan\theta = \sqrt{3}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = \frac{\pi}{3}$$

$$\sec\left(\tan^{-1}\sqrt{3}\right) = \sec\frac{\pi}{3} = 2$$

17. $\sin\left[\tan^{-1}(-1)\right]$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals -1.

$$\tan\theta = -1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{4}$$

$$\sin\left[\tan^{-1}(-1)\right] = \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

18. $\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta = -\frac{\sqrt{3}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{3}$$

$$\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

19. $\sec\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{6}$$

$$\sec\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = \sec\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$$

20. $\csc\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

$$\cos\theta = -\frac{\sqrt{3}}{2} \quad 0 \le \theta \le \pi$$

$$\theta = \frac{5\pi}{6}$$

$$\csc\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \csc\frac{5\pi}{6} = 2$$

21. $\cos^{-1}\left(\sin\frac{5\pi}{4}\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{2}}{2}$.

$$\cos\theta = -\frac{\sqrt{2}}{2}, \quad 0 \le \theta \le \pi$$

$$\theta = \frac{3\pi}{4}$$

$$\cos^{-1}\left(\sin\frac{5\pi}{4}\right) = \frac{3\pi}{4}$$

22. $\tan^{-1}\left(\cot\frac{2\pi}{3}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $-\frac{1}{\sqrt{3}}$.

$$\tan\theta = -\frac{1}{\sqrt{3}}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{6}$$

$$\tan^{-1}\left(\cot\frac{2\pi}{3}\right) = -\frac{\pi}{6}$$

23. $\sin^{-1}\left[\cos\left(-\frac{7\pi}{6}\right)\right] = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta = -\frac{\sqrt{3}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{3}$$
$$\sin^{-1}\left[\cos\left(-\frac{7\pi}{6}\right)\right] = -\frac{\pi}{3}$$

24. $\cos^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right] = \cos^{-1}(-1)$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals -1.

$$\cos\theta = -1, \quad 0 \le \theta \le \pi$$
$$\theta = \frac{\pi}{3}$$
$$\cos^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right] = \pi$$

25. $\tan\left(\sin^{-1}\frac{1}{3}\right)$

Let $\theta = \sin^{-1}\frac{1}{3}$. Since $\sin\theta = \frac{1}{3}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant I, and we let $y = 1$ and $r = 3$.

Solve for x:

$x^2 + 1 = 9$

$x^2 = 8$

$x = \pm\sqrt{8} = \pm 2\sqrt{2}$

Since θ is in quadrant I, $x = 2\sqrt{2}$.

$$\tan\left(\sin^{-1}\frac{1}{3}\right) = \tan\theta = \frac{y}{x} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

26. $\tan\left(\cos^{-1}\frac{1}{3}\right)$

Let $\theta = \cos^{-1}\frac{1}{3}$. Since $\cos\theta = \frac{1}{3}$ and $0 \le \theta \le \pi$, θ is in quadrant I, and we let $x = 1$ and $r = 3$.

Solve for y:

$1 + y^2 = 9$

$y^2 = 8$

$y = \pm\sqrt{8} = \pm 2\sqrt{2}$

Since θ is in quadrant I, $y = 2\sqrt{2}$.

$$\tan\left(\cos^{-1}\frac{1}{3}\right) = \tan\theta = \frac{y}{x} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

27. $\sec\left(\tan^{-1}\frac{1}{2}\right)$

Let $\theta = \tan^{-1}\frac{1}{2}$. Since $\tan\theta = \frac{1}{2}$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, θ is in quadrant I, and we let $x = 2$ and $y = 1$.

Solve for r:

$2^2 + 1 = r^2$

$r^2 = 5$

$r = \sqrt{5}$

θ is in quadrant I.

$$\sec\left(\tan^{-1}\frac{1}{2}\right) = \sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{2}$$

28. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{3}\right)$

Let $\theta = \sin^{-1}\frac{\sqrt{2}}{3}$. Since $\sin\theta = \frac{\sqrt{2}}{3}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant I, and we let $y = \sqrt{2}$ and $r = 3$.

Solve for x:

$x^2 + 2 = 9$

$x^2 = 7$

$x = \pm\sqrt{7}$

Since θ is in quadrant I, $x = \sqrt{7}$.

$$\cos\left(\sin^{-1}\frac{\sqrt{2}}{3}\right) = \cos\theta = \frac{x}{r} = \frac{\sqrt{7}}{3}$$

29. $\cot\left[\sin^{-1}\left(-\frac{\sqrt{2}}{3}\right)\right]$

Let $\theta = \sin^{-1}\left(-\frac{\sqrt{2}}{3}\right)$. Since $\sin\theta = -\frac{\sqrt{2}}{3}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant IV, and we let $y = -\sqrt{2}$ and $r = 3$.

Solve for x:

$x^2+2=9$
$x^2=7$
$x=\pm\sqrt{7}$
Since θ is in quadrant IV, $x=\sqrt{7}$.
$\cot\left[\sin^{-1}\left(-\frac{\sqrt{2}}{3}\right)\right]=\cot\theta=\frac{x}{y}=\frac{\sqrt{7}}{-\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=-\frac{\sqrt{14}}{2}$

30. $\csc\left[\tan^{-1}(-2)\right]$
Let $\theta=\tan^{-1}(-2)$. Since $\tan\theta=-2$ and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, θ is in quadrant IV, and we let $x=1$ and $y=-2$.
Solve for r:
$1+4=r^2$
$r^2=5$
$r=\pm\sqrt{5}$
Since θ is in quadrant IV, $r=\sqrt{5}$.
$\csc\left[\tan^{-1}(-2)\right]=\csc\theta=\frac{r}{y}=\frac{\sqrt{5}}{-2}=-\frac{\sqrt{5}}{2}$

31. $\sin\left[\tan^{-1}(-3)\right]$
Let $\theta=\tan^{-1}(-3)$. Since $\tan\theta=-3$ and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, θ is in quadrant IV, and we let $x=1$ and $y=-3$.
Solve for r:
$1+9=r^2$
$r^2=10$
$r=\pm\sqrt{10}$
Since θ is in quadrant IV, $r=\sqrt{10}$.
$\sin\left[\tan^{-1}(-3)\right]=\sin\theta=\frac{y}{r}$
$=\frac{-3}{\sqrt{10}}\cdot\frac{\sqrt{10}}{\sqrt{10}}=-\frac{3\sqrt{10}}{10}$

32. $\cot\left[\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right]$
Let $\theta=\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)$. Since $\cos\theta=-\frac{\sqrt{3}}{3}$ and $0\le\theta\le\pi$, θ is in quadrant II, and we let $x=-\sqrt{3}$ and $r=3$.
Solve for y:
$3+y^2=9$
$y^2=6$
$y=\pm\sqrt{6}$
Since θ is in quadrant II, $y=\sqrt{6}$.
$\cot\left[\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right]=\cot\theta=\frac{x}{y}$
$=\frac{-\sqrt{3}}{\sqrt{6}}=\frac{-1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$

33. $\sec\left(\sin^{-1}\frac{2\sqrt{5}}{5}\right)$
Let $\theta=\sin^{-1}\frac{2\sqrt{5}}{5}$. Since $\sin\theta=\frac{2\sqrt{5}}{5}$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, θ is in quadrant I, and we let $y=2\sqrt{5}$ and $r=5$.
Solve for x:
$x^2+20=25$
$x^2=5$
$x=\pm\sqrt{5}$
Since θ is in quadrant I, $x=\sqrt{5}$.
$\sec\left(\sin^{-1}\frac{2\sqrt{5}}{5}\right)=\sec\theta=\frac{r}{x}=\frac{5}{\sqrt{5}}=\sqrt{5}$

34. $\csc\left(\tan^{-1}\frac{1}{2}\right)$
Let $\theta=\tan^{-1}\frac{1}{2}$. Since $\tan\theta=\frac{1}{2}$ and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, θ is in quadrant I, and we let $x=2$ and $y=1$.
Solve for r:
$2^2+1=r^2$
$r^2=5$
$r=\sqrt{5}$
θ is in quadrant I.
$\csc\left(\tan^{-1}\frac{1}{2}\right)=\csc\theta=\frac{r}{y}=\frac{\sqrt{5}}{1}=\sqrt{5}$

35. $\sin^{-1}\left(\cos\frac{3\pi}{4}\right)=\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$

36. $\cos^{-1}\left(\sin\frac{7\pi}{6}\right)=\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

37. $\cot^{-1}\sqrt{3}$

We are finding the angle $\theta,\ 0<\theta<\pi,$ whose cotangent equals $\sqrt{3}$.

$\cot\theta=\sqrt{3},\quad 0<\theta<\pi$

$\theta=\frac{\pi}{6}$

$\cot^{-1}\sqrt{3}=\frac{\pi}{6}$

38. $\cot^{-1}1$

We are finding the angle $\theta,\ 0<\theta<\pi,$ whose cotangent equals 1.

$\cot\theta=1,\quad 0<\theta<\pi$

$\theta=\frac{\pi}{4}$

$\cot^{-1}1=\frac{\pi}{4}$

39. $\csc^{-1}(-1)$

We are finding the angle $\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, $\theta\ne 0$, whose cosecant equals -1.

$\csc\theta=-1,\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ \theta\ne 0$

$\theta=-\frac{\pi}{2}$

$\csc^{-1}(-1)=-\frac{\pi}{2}$

40. $\csc^{-1}\sqrt{2}$

We are finding the angle $\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, $\theta\ne 0$, whose cosecant equals $\sqrt{2}$.

$\csc\theta=\sqrt{2},\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ \theta\ne 0$

$\theta=\frac{\pi}{4}$

$\csc^{-1}\sqrt{2}=\frac{\pi}{4}$

41. $\sec^{-1}\frac{2\sqrt{3}}{3}$

We are finding the angle $\theta,\ 0\le\theta\le\pi$, $\theta\ne\frac{\pi}{2}$, whose secant equals $\frac{2\sqrt{3}}{3}$.

$\sec\theta=\frac{2\sqrt{3}}{3},\quad 0\le\theta\le\pi,\ \theta\ne\frac{\pi}{2}$

$\theta=\frac{\pi}{6}$

$\sec^{-1}\frac{2\sqrt{3}}{3}=\frac{\pi}{6}$

42. $\sec^{-1}(-2)$

We are finding the angle $\theta,\ 0\le\theta\le\pi$, $\theta\ne\frac{\pi}{2}$, whose secant equals -2.

$\sec\theta=-2,\quad 0\le\theta\le\pi,\ \theta\ne\frac{\pi}{2}$

$\theta=\frac{2\pi}{3}$

$\sec^{-1}(-2)=\frac{2\pi}{3}$

43. $\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

We are finding the angle $\theta,\ 0<\theta<\pi,$ whose cotangent equals $-\frac{\sqrt{3}}{3}$.

$\cot\theta=-\frac{\sqrt{3}}{3},\quad 0<\theta<\pi$

$\theta=\frac{2\pi}{3}$

$\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)=\frac{2\pi}{3}$

44. $\csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

We are finding the angle $\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$,

$\theta \neq 0$, whose cosecant equals $-\frac{2\sqrt{3}}{3}$.

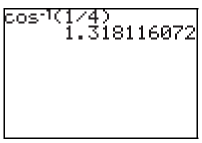
$$\csc\theta = -\frac{2\sqrt{3}}{3}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ \theta \neq 0$$

$$\theta = -\frac{\pi}{3}$$

$$\csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = -\frac{\pi}{3}$$

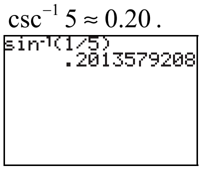
45. $\sec^{-1} 4 = \cos^{-1}\frac{1}{4}$

We seek the angle θ, $0 \le \theta \le \pi$, whose cosine equals $\frac{1}{4}$. Now $\cos\theta = \frac{1}{4}$, so θ lies in quadrant I. The calculator yields $\cos^{-1}\frac{1}{4} \approx 1.32$, which is an angle in quadrant I, so $\sec^{-1}(4) \approx 1.32$.



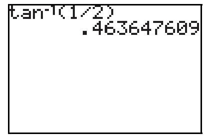
46. $\csc^{-1} 5 = \sin^{-1}\frac{1}{5}$

We seek the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{1}{5}$. Now $\sin\theta = \frac{1}{5}$, so θ lies in quadrant I. The calculator yields $\sin^{-1}\frac{1}{5} \approx 0.20$, which is an angle in quadrant I, so $\csc^{-1} 5 \approx 0.20$.



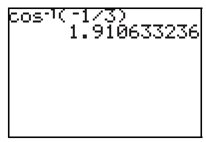
47. $\cot^{-1} 2 = \tan^{-1}\frac{1}{2}$

We seek the angle θ, $0 \le \theta \le \pi$, whose tangent equals $\frac{1}{2}$. Now $\tan\theta = \frac{1}{2}$, so θ lies in quadrant I. The calculator yields $\text{an}^{-1}\frac{1}{2} \approx 0.46$, which is an angle in quadrant I, so $\cot^{-1}(2) \approx 0.46$.



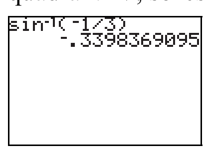
48. $\sec^{-1}(-3) = \cos^{-1}\left(-\frac{1}{3}\right)$

We seek the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{1}{3}$. Now $\cos\theta = -\frac{1}{3}$, θ lies in quadrant II. The calculator yields $\cos^{-1}\left(-\frac{1}{3}\right) \approx 1.91$, which is an angle in quadrant II, so $\sec^{-1}(-3) \approx 1.91$.



49. $\csc^{-1}(-3) = \sin^{-1}\left(-\frac{1}{3}\right)$

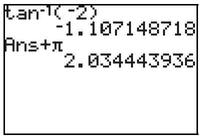
We seek the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{3}$. Now $\sin\theta = -\frac{1}{3}$, so θ lies in quadrant IV. The calculator yields $\sin^{-1}\left(-\frac{1}{3}\right) \approx -0.34$, which is an angle in quadrant IV, so $\csc^{-1}(-3) \approx -0.34$.



50. $\cot^{-1}\left(-\frac{1}{2}\right) = \tan^{-1}(-2)$

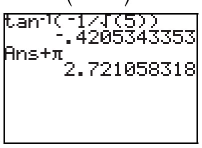
We seek the angle θ, $0 \le \theta \le \pi$, whose tangent equals -2. Now $\tan\theta = -2$, so θ lies in quadrant II. The calculator yields $\tan^{-1}(-2) \approx -1.11$, which is an angle in quadrant IV. Since θ lies in quadrant II, $\theta \approx -1.11 + \pi \approx 2.03$. Therefore,

$\cot^{-1}\left(-\frac{1}{2}\right) \approx 2.03$.



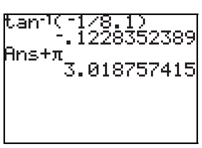
51. $\cot^{-1}\left(-\sqrt{5}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{5}}\right)$

We seek the angle θ, $0 \le \theta \le \pi$, whose tangent equals $-\frac{1}{\sqrt{5}}$. Now $\tan\theta = -\frac{1}{\sqrt{5}}$, so θ lies in quadrant II. The calculator yields $\tan^{-1}\left(-\frac{1}{\sqrt{5}}\right) \approx -0.42$, which is an angle in quadrant IV. Since θ is in quadrant II, $\theta \approx -0.42 + \pi \approx 2.72$. Therefore, $\cot^{-1}\left(-\sqrt{5}\right) \approx 2.72$.



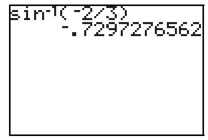
52. $\cot^{-1}(-8.1) = \tan^{-1}\left(-\frac{1}{8.1}\right)$

We seek the angle θ, $0 \le \theta \le \pi$, whose tangent equals $-\frac{1}{8.1}$. Now $\tan\theta = -\frac{1}{8.1}$, so θ lies in quadrant II. The calculator yields $\tan^{-1}\left(-\frac{1}{8.1}\right) \approx -0.12$, which is an angle in quadrant IV. Since θ is in quadrant II, $\theta \approx -0.12 + \pi \approx 3.02$. Thus, $\cot^{-1}(-8.1) \approx 3.02$.



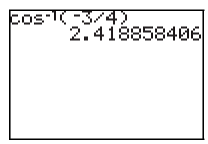
53. $\csc^{-1}\left(-\frac{3}{2}\right) = \sin^{-1}\left(-\frac{2}{3}\right)$

We seek the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$, whose sine equals $-\frac{2}{3}$. Now $\sin\theta = -\frac{2}{3}$, so θ lies in quadrant IV. The calculator yields $\sin^{-1}\left(-\frac{2}{3}\right) \approx -0.73$, which is an angle in quadrant IV, so $\csc^{-1}\left(-\frac{3}{2}\right) \approx -0.73$.



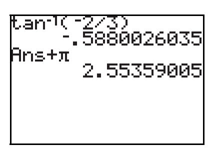
54. $\sec^{-1}\left(-\frac{4}{3}\right) = \cos^{-1}\left(-\frac{3}{4}\right)$

We are finding the angle θ, $0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$, whose cosine equals $-\frac{3}{4}$. Now $\cos\theta = -\frac{3}{4}$, so θ lies in quadrant II. The calculator yields $\cos^{-1}\left(-\frac{3}{4}\right) \approx 2.42$, which is an angle in quadrant II, so $\sec^{-1}\left(-\frac{4}{3}\right) \approx 2.42$.



55. $\cot^{-1}\left(-\frac{3}{2}\right) = \tan^{-1}\left(-\frac{2}{3}\right)$

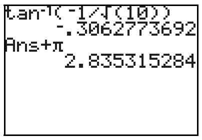
We are finding the angle θ, $0 \le \theta \le \pi$, whose tangent equals $-\frac{2}{3}$. Now $\tan\theta = -\frac{2}{3}$, so θ lies in quadrant II. The calculator yields $\tan^{-1}\left(-\frac{2}{3}\right) \approx -0.59$, which is an angle in quadrant IV. Since θ is in quadrant II, $\theta \approx -0.59 + \pi \approx 2.55$. Thus, $\cot^{-1}\left(-\frac{3}{2}\right) \approx 2.55$.



56. $\cot^{-1}\left(-\sqrt{10}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{10}}\right)$

We are finding the angle θ, $0 \le \theta \le \pi$, whose tangent equals $-\frac{1}{\sqrt{10}}$. Now $\tan\theta = -\frac{1}{\sqrt{10}}$, so θ lies in quadrant II. The calculator yields

$\tan^{-1}\left(-\frac{1}{\sqrt{10}}\right) \approx -0.306$, which is an angle in quadrant IV. Since θ is in quadrant II, $\theta \approx -0.306 + \pi \approx 2.84$. So, $\cot^{-1}\left(-\sqrt{10}\right) \approx 2.84$.



57. Let $\theta = \tan^{-1} u$ so that $\tan\theta = u$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $-\infty < u < \infty$. Then,

$$\cos\left(\tan^{-1} u\right) = \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{\sec^2\theta}} = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+u^2}}$$

58. Let $\theta = \cos^{-1} u$ so that $\cos\theta = u$, $0 \le \theta \le \pi$, $-1 \le u \le 1$. Then,

$$\sin\left(\cos^{-1} u\right) = \sin\theta = \sqrt{\sin^2\theta} = \sqrt{1-\cos^2\theta} = \sqrt{1-u^2}$$

59. Let $\theta = \sin^{-1} u$ so that $\sin\theta = u$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $-1 \le u \le 1$. Then,

$$\tan\left(\sin^{-1} u\right) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{\cos^2\theta}} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{u}{\sqrt{1-u^2}}$$

60. Let $\theta = \cos^{-1} u$ so that $\cos\theta = u$, $0 \le \theta \le \pi$, $-1 \le u \le 1$. Then,

$$\tan\left(\cos^{-1} u\right) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{\sin^2\theta}}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-u^2}}{u}$$

61. Let $\theta = \sec^{-1} u$ so that $\sec\theta = u$, $0 \le \theta \le \pi$ and $\theta \ne \frac{\pi}{2}$, $|u| \ge 1$. Then,

$$\sin\left(\sec^{-1} u\right) = \sin\theta = \sqrt{\sin^2\theta} = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{\sec^2\theta}} = \frac{\sqrt{\sec^2\theta-1}}{\sqrt{\sec^2\theta}} = \frac{\sqrt{u^2-1}}{|u|}$$

62. Let $\theta = \cot^{-1} u$ so that $\cot\theta = u$, $0 < \theta < \pi$, $-\infty < u < \infty$. Then,

$$\sin\left(\cot^{-1} u\right) = \sin\theta = \sqrt{\sin^2\theta} = \frac{1}{\sqrt{\csc^2\theta}} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+u^2}}$$

63. Let $\theta = \csc^{-1} u$ so that $\csc\theta = u$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $|u| \ge 1$. Then,

$$\cos\left(\csc^{-1} u\right) = \cos\theta = \cos\theta \cdot \frac{\sin\theta}{\sin\theta} = \cot\theta\sin\theta = \frac{\cot\theta}{\csc\theta} = \frac{\sqrt{\cot^2\theta}}{\csc\theta} = \frac{\sqrt{\csc^2\theta-1}}{\csc\theta} = \frac{\sqrt{u^2-1}}{|u|}$$

64. Let $\theta = \sec^{-1} u$ so that $\sec\theta = u$, $0 \le \theta \le \pi$ and $\theta \ne \frac{\pi}{2}$, $|u| \ge 1$. Then,

$$\cos\left(\sec^{-1} u\right) = \cos\theta = \frac{1}{\sec\theta} = \frac{1}{u}$$

65. Let $\theta = \cot^{-1} u$ so that $\cot\theta = u$, $0 < \theta < \pi$, $-\infty < u < \infty$. Then,

$$\tan\left(\cot^{-1} u\right) = \tan\theta = \frac{1}{\cot\theta} = \frac{1}{u}$$

66. Let $\theta = \sec^{-1} u$ so that $\sec\theta = u$, $0 \le \theta \le \pi$ and $\theta \ne \frac{\pi}{2}$, $|u| \ge 1$. Then,

$$\tan\left(\sec^{-1} u\right) = \tan\theta = \sqrt{\tan^2\theta} = \sqrt{\sec^2\theta-1} = \sqrt{u^2-1}$$

67. $g\left(f^{-1}\left(\frac{12}{13}\right)\right)=\cos\left(\sin^{-1}\frac{12}{13}\right)$

Let $\theta=\sin^{-1}\frac{12}{13}$. Since $\sin\theta=\frac{12}{13}$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, θ is in quadrant I, and we let $y=12$ and $r=13$. Solve for *x*:

$$x^2+12^2=13^2$$
$$x^2+144=169$$
$$x^2=25$$
$$x=\pm\sqrt{25}=\pm 5$$

Since θ is in quadrant I, $x=5$.

$$g\left(f^{-1}\left(\frac{12}{13}\right)\right)=\cos\left(\sin^{-1}\frac{12}{13}\right)=\cos\theta=\frac{x}{r}=\frac{5}{13}$$

68. $f\left(g^{-1}\left(\frac{5}{13}\right)\right)=\sin\left(\cos^{-1}\frac{5}{13}\right)$

Let $\theta=\cos^{-1}\frac{5}{13}$. Since $\cos\theta=\frac{5}{13}$ and $0\le\theta\le\pi$, θ is in quadrant I, and we let $x=5$ and $r=13$. Solve for *y*:

$$5^2+y^2=13^2$$
$$25+y^2=169$$
$$y^2=144$$
$$y=\pm\sqrt{144}=\pm 12$$

Since θ is in quadrant I, $y=12$.

$$f\left(g^{-1}\left(\frac{5}{13}\right)\right)=\sin\left(\cos^{-1}\frac{5}{13}\right)=\sin\theta=\frac{y}{r}=\frac{12}{13}$$

69. $g^{-1}\left(f\left(\frac{7\pi}{4}\right)\right)=\cos^{-1}\left(\sin\frac{7\pi}{4}\right)$

$$=\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)=\frac{3\pi}{4}$$

70. $f^{-1}\left(g\left(\frac{5\pi}{6}\right)\right)=\sin^{-1}\left(\cos\frac{5\pi}{6}\right)$

$$=\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$$

71. $h\left(f^{-1}\left(-\frac{3}{5}\right)\right)=\tan\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

Let $\theta=\sin^{-1}\left(-\frac{3}{5}\right)$. Since $\sin\theta=-\frac{3}{5}$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, θ is in quadrant IV, and we let $y=-3$ and $r=5$. Solve for *x*:

$$x^2+(-3)^2=5^2$$
$$x^2+9=25$$
$$x^2=16$$
$$x=\pm\sqrt{16}=\pm 4$$

Since θ is in quadrant IV, $x=4$.

$$h\left(f^{-1}\left(-\frac{3}{5}\right)\right)=\tan\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$$
$$=\tan\theta=\frac{y}{x}=\frac{-3}{4}=-\frac{3}{4}$$

72. $h\left(g^{-1}\left(-\frac{4}{5}\right)\right)=\tan\left(\cos^{-1}\left(-\frac{4}{5}\right)\right)$

Let $\theta=\cos^{-1}\left(-\frac{4}{5}\right)$. Since $\cos\theta=-\frac{4}{5}$ and $0\le\theta\le\pi$, θ is in quadrant II, and we let $x=-4$ and $r=5$. Solve for *y*:

$$(-4)^2+y^2=5^2$$
$$16+y^2=25$$
$$y^2=9$$
$$y=\pm\sqrt{9}=\pm 3$$

Since θ is in quadrant II, $y=3$.

$$h\left(g^{-1}\left(-\frac{4}{5}\right)\right)=\tan\left(\cos^{-1}\left(-\frac{4}{5}\right)\right)$$
$$=\tan\theta=\frac{y}{x}=\frac{-3}{4}=-\frac{3}{4}$$

73. $g\left(h^{-1}\left(\frac{12}{5}\right)\right)=\cos\left(\tan^{-1}\frac{12}{5}\right)$

Let $\theta=\tan^{-1}\frac{12}{5}$. Since $\tan\theta=\frac{12}{5}$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, θ is in quadrant I, and we let $x=5$ and $y=12$. Solve for *r*:

$r^2 = 5^2 + 12^2$

$r^2 = 25 + 144 = 169$

$r = \pm\sqrt{169} = \pm 13$

Now, r must be positive, so $r = 13$.

$g\left(h^{-1}\left(\frac{12}{5}\right)\right) = \cos\left(\tan^{-1}\frac{12}{5}\right) = \cos\theta = \frac{x}{r} = \frac{5}{13}$

74. $f\left(h^{-1}\left(\frac{5}{12}\right)\right) = \sin\left(\tan^{-1}\frac{5}{12}\right)$

Let $\theta = \tan^{-1}\frac{5}{12}$. Since $\tan\theta = \frac{5}{12}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant I, and we let $x = 12$ and $y = 5$. Solve for r:

$r^2 = 12^2 + 5^2$

$r^2 = 144 + 25 = 169$

$r = \pm\sqrt{169} = \pm 13$

Now, r must be positive, so $r = 13$.

$f\left(h^{-1}\left(\frac{5}{12}\right)\right) = \sin\left(\tan^{-1}\frac{5}{12}\right) = \sin\theta = \frac{y}{r} = \frac{5}{13}$

75. $g^{-1}\left(f\left(-\frac{4\pi}{3}\right)\right) = \cos^{-1}\left(\sin\left(-\frac{4\pi}{3}\right)\right)$

$= \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

76. $g^{-1}\left(f\left(-\frac{5\pi}{6}\right)\right) = \cos^{-1}\left(\sin\left(-\frac{5\pi}{6}\right)\right)$

$= \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

77. $h\left(g^{-1}\left(-\frac{1}{4}\right)\right) = \tan\left(\cos^{-1}\left(-\frac{1}{4}\right)\right)$

Let $\theta = \cos^{-1}\left(-\frac{1}{4}\right)$. Since $\cos\theta = -\frac{1}{4}$ and $0 \le \theta \le \pi$, θ is in quadrant II, and we let $x = -1$ and $r = 4$. Solve for y:

$(-1)^2 + y^2 = 4^2$

$1 + y^2 = 16$

$y^2 = 15$

$y = \pm\sqrt{15}$

Since θ is in quadrant II, $y = \sqrt{15}$.

$h\left(g^{-1}\left(-\frac{1}{4}\right)\right) = \tan\left(\cos^{-1}\left(-\frac{1}{4}\right)\right)$

$= \tan\theta = \frac{y}{x} = \frac{\sqrt{15}}{-1} = -\sqrt{15}$

78. $h\left(f^{-1}\left(-\frac{2}{5}\right)\right) = \tan\left(\sin^{-1}\left(-\frac{2}{5}\right)\right)$

Let $\theta = \sin^{-1}\left(-\frac{2}{5}\right)$. Since $\sin\theta = -\frac{2}{5}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant IV, and we let $y = -2$ and $r = 5$. Solve for x:

$x^2 + (-2)^2 = 5^2$

$x^2 + 4 = 25$

$x^2 = 21$

$x = \pm\sqrt{21}$

Since θ is in quadrant IV, $x = \sqrt{21}$.

$h\left(f^{-1}\left(-\frac{2}{5}\right)\right) = \tan\left(\sin^{-1}\left(-\frac{2}{5}\right)\right)$

$= \tan\theta = \frac{y}{x} = \frac{-2}{\sqrt{21}} = -\frac{2\sqrt{21}}{21}$

79. a. Since the diameter of the base is 45 feet, we have $r = \frac{45}{2} = 22.5$ feet. Thus,

$\theta = \cot^{-1}\left(\frac{22.5}{14}\right) = 31.89°$.

b. $\theta = \cot^{-1}\frac{r}{h}$

$\cot\theta = \frac{r}{h} \quad \rightarrow \quad r = h\cot\theta$

Here we have $\theta = 31.89°$ and $h = 17$ feet. Thus, $r = 17\cot(31.89°) = 27.32$ feet and the diameter is $2(27.32) = 54.64$ feet.

c. From part (b), we get $h = \frac{r}{\cot\theta}$.

The radius is $\frac{122}{2} = 61$ feet.

$h = \frac{r}{\cot\theta} = \frac{61}{22.5/14} \approx 37.96$ feet.

Thus, the height is 37.96 feet.

80. a. Since the diameter of the base is 6.68 feet, we have $r = \frac{6.68}{2} = 3.34$ feet. Thus,

$$\theta = \cot^{-1}\left(\frac{3.34}{4}\right) = 50.14°$$

b. $\theta = \cot^{-1}\frac{r}{h}$

$\cot\theta = \frac{r}{h} \quad \rightarrow \quad h = \frac{r}{\cot\theta}$

Here we have $\theta = 50.14°$ and $r = 4$ feet.

Thus, $h = \frac{4}{\cot(50.14°)} = 4.79$ feet. The bunker will be 4.79 feet high.

c. $\theta_{TG} = \cot^{-1}\left(\frac{4.22}{6}\right) = 54.88°$

From part (a) we have $\theta_{USGA} = 50.14°$. For steep bunkers, a larger angle of repose is required. Therefore, the Tour Grade 50/50 sand is better suited since it has a larger angle of repose.

81. a. $\cot\theta = \frac{2x}{2y + gt^2}$

$$\theta = \cot^{-1}\left(\frac{2x}{2y + gt^2}\right)$$

The artillery shell begins at the origin and lands at the coordinates $(6175, 2450)$. Thus,

$$\theta = \cot^{-1}\left(\frac{2 \cdot 6175}{2 \cdot 2450 + 32.2(2.27)^2}\right)$$

$$\approx \cot^{-1}(2.437858) \approx 22.3°$$

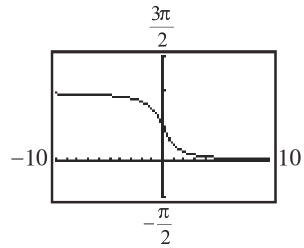
The artilleryman used an angle of elevation of $22.3°$.

b. $\sec\theta = \frac{v_0 t}{x}$

$$v_0 = \frac{x\sec\theta}{t} = \frac{(6175)\sec(22.3°)}{2.27}$$

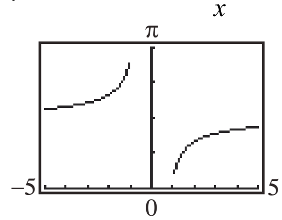
$$= 2940.23 \text{ ft/sec}$$

82. Let. $y = \cot^{-1} x = \cos^{-1}\frac{x}{\sqrt{x^2+1}}$

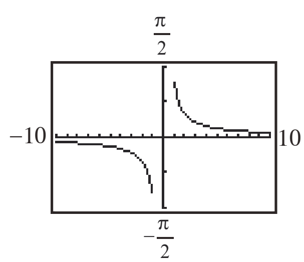


Note that the range of $y = \cot^{-1} x$ is $(0, \pi)$, so $\tan^{-1}\frac{1}{x}$ will not work.

83. $y = \sec^{-1} x = \cos^{-1}\frac{1}{x}$



84. $y = \csc^{-1} x = \sin^{-1}\frac{1}{x}$



85 – 86. Answers will vary.

87. $f(x) = 4x^4 + 21x^2 - 100$

$4x^4 + 21x^2 - 100 = 0$

$(x^2 - 4)(x^2 + 25) = 0$

$x^2 - 4 = 0 \quad \text{or} \quad x^2 + 25 = 0$

$x = \pm 2 \quad \text{or} \quad x = \pm 5i$

So the solution set is: $\{-2, 2, -5i, 5i\}$

88. $f(-x) = (-x)^3 + (-x)^2 - (-x)$

$= -x^3 + x^2 + x \neq f(x)$

So the function is not even.

$f(-x) = (-x)^3 + (-x)^2 - (-x)$

$= -(x^3 - x^2 - x) \neq -f(x)$

So the function is not odd.

89. $315\left(\frac{\pi}{180}\right)=\frac{7\pi}{4}$ radians

90. $75°=\frac{5\pi}{12}$

$s=r\theta$

$=6\left(\frac{5\pi}{12}\right)$

$=\frac{5\pi}{2}\approx 7.85$ in.

Section 7.3

1. $3x-5=-x+1$

$4x=6$

$x=\frac{6}{4}=\frac{3}{2}$

The solution set is $\left\{\frac{3}{2}\right\}$.

2. $\frac{\sqrt{2}}{2},\ -\frac{1}{2}$

3. $4x^2-x-5=0$

$(4x-5)(x+1)=0$

$4x-5=0$ or $x+1=0$

$x=\frac{5}{4}$ or $x=-1$

The solution set is $\left\{-1,\ \frac{5}{4}\right\}$.

4. $x^2-x-1=0$

$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$

$=\frac{1\pm\sqrt{1+4}}{2}$

$=\frac{1\pm\sqrt{5}}{2}$

The solution set is $\left\{\frac{1-\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2}\right\}$.

5. $(2x-1)^2-3(2x-1)-4=0$

$[(2x-1)+1][(2x-1)-4]=0$

$2x(2x-5)=0$

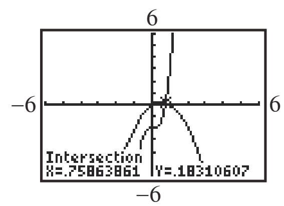
$2x=0$ or $2x-5=0$

$x=0$ or $x=\frac{5}{2}$

The solution set is $\left\{0,\ \frac{5}{2}\right\}$.

6. $5x^3-2=x-x^2$

Let $y_1=5x^3-2$ and $y_2=x-x^2$. Use INTERSECT to find the solution(s):



In this case, the graphs only intersect in one location, so the equation has only one solution. Rounding as directed, the solutions set is $\{0.76\}$.

7. False because of the circular nature of the functions.

8. True

9. True

10. False, 2 is outside the range of the sin function.

11. d

12. a

13. $2\sin\theta+3=2$

$2\sin\theta=-1$

$\sin\theta=-\frac{1}{2}$

$\theta=\frac{7\pi}{6}+2k\pi$ or $\theta=\frac{11\pi}{6}+2k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{7\pi}{6},\ \frac{11\pi}{6}\right\}$.

14. $1-\cos\theta=\frac{1}{2}$

$1-\cos\theta=\frac{1}{2}$

$\frac{1}{2}=\cos\theta$

$\theta=\frac{\pi}{3}+2k\pi$ or $\theta=\frac{5\pi}{3}+2k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{\pi}{3},\frac{5\pi}{3}\right\}$.

15. $2\sin\theta+1=0$

$2\sin\theta=-1$

$\sin\theta=-\frac{1}{2}$

$\theta=\frac{7\pi}{6}+2k\pi$ or $\theta=\frac{11\pi}{6}+2k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{7\pi}{6},\frac{11\pi}{6}\right\}$.

16. $\cos\theta+1=0$

$\cos\theta=-1$

$\theta=\pi+2k\pi$, k is any integer

On the interval $0\le\theta<2\pi$, the solution set is $\{\pi\}$.

17. $\tan\theta+1=0$

$\tan\theta=-1$

$\theta=\frac{3\pi}{4}+k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{3\pi}{4},\frac{7\pi}{4}\right\}$.

18. $\sqrt{3}\cot\theta+1=0$

$\sqrt{3}\cot\theta=-1$

$\cot\theta=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$

$\theta=\frac{2\pi}{3}+k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{2\pi}{3},\frac{5\pi}{3}\right\}$.

19. $4\sec\theta+6=-2$

$4\sec\theta=-8$

$\sec\theta=-2$

$\theta=\frac{2\pi}{3}+2k\pi$ or $\theta=\frac{4\pi}{3}+2k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{2\pi}{3},\frac{4\pi}{3}\right\}$.

20. $5\csc\theta-3=2$

$5\csc\theta=5$

$\csc\theta=1$

$\theta=\frac{\pi}{2}+2k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{\pi}{2}\right\}$.

21. $3\sqrt{2}\cos\theta+2=-1$

$3\sqrt{2}\cos\theta=-3$

$\cos\theta=-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$

$\theta=\frac{3\pi}{4}+2k\pi$ or $\theta=\frac{5\pi}{4}+2k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{3\pi}{4},\frac{5\pi}{4}\right\}$.

22. $4\sin\theta+3\sqrt{3}=\sqrt{3}$

$4\sin\theta=-2\sqrt{3}$

$\sin\theta=-\frac{2\sqrt{3}}{4}=-\frac{\sqrt{3}}{2}$

$\theta=\frac{4\pi}{3}+2k\pi$ or $\theta=\frac{5\pi}{3}+2k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{4\pi}{3},\frac{5\pi}{3}\right\}$.

23. $4\cos^2\theta=1$

$\cos^2\theta=\frac{1}{4}$

$\cos\theta=\pm\frac{1}{2}$

$\theta=\frac{\pi}{3}+k\pi$ or $\theta=\frac{2\pi}{3}+k\pi$, k is any integer

On the interval $0\le\theta<2\pi$, the solution set is $\left\{\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}\right\}$.

24. $\tan^2\theta = \frac{1}{3}$

$\tan\theta = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$

$\theta = \frac{\pi}{6} + k\pi$ or $\theta = \frac{5\pi}{6} + k\pi$, k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

25. $2\sin^2\theta - 1 = 0$

$2\sin^2\theta = 1$

$\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{4} + k\pi$ or $\theta = \frac{3\pi}{4} + k\pi$, k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$.

26. $4\cos^2\theta - 3 = 0$

$4\cos^2\theta = 3$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6} + k\pi$ or $\theta = \frac{5\pi}{6} + k\pi$, k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

27. $\sin(3\theta) = -1$

$3\theta = \frac{3\pi}{2} + 2k\pi$

$\theta = \frac{\pi}{2} + \frac{2k\pi}{3}$, k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

28. $\tan\left(\frac{\theta}{2}\right) = \sqrt{3}$

$\frac{\theta}{2} = \frac{\pi}{3} + k\pi$, k is any integer

$\theta = \frac{2\pi}{3} + 2\pi k$, k is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{2\pi}{3}\right\}$.

29. $\cos(2\theta) = -\frac{1}{2}$

$2\theta = \frac{2\pi}{3} + 2k\pi$ or $2\theta = \frac{4\pi}{3} + 2k\pi$

$\theta = \frac{\pi}{3} + k\pi$ or $\theta = \frac{2\pi}{3} + k\pi$, k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

30. $\tan(2\theta) = -1$

$2\theta = \frac{3\pi}{4} + k\pi$, k is any integer

$\theta = \frac{3\pi}{8} + \frac{k\pi}{2}$, k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$.

31. $\sec\frac{3\theta}{2} = -2$

$\frac{3\theta}{2} = \frac{2\pi}{3} + 2k\pi$ or $\frac{3\theta}{2} = \frac{4\pi}{3} + 2k\pi$

$\theta = \frac{4\pi}{9} + \frac{4k\pi}{3}$ or $\theta = \frac{8\pi}{9} + \frac{4k\pi}{3}$,

k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}\right\}$.

32. $\cot\frac{2\theta}{3} = -\sqrt{3}$

$\frac{2\theta}{3} = \frac{5\pi}{6} + k\pi$, k is any integer

$\theta = \frac{5\pi}{4} + \frac{3k\pi}{2}$, k is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{5\pi}{4}\right\}$.

33. $\cos\left(2\theta-\frac{\pi}{2}\right)=-1$

$2\theta-\frac{\pi}{2}=\pi+2k\pi$

$2\theta=\frac{3\pi}{2}+2k\pi$

$\theta=\frac{3\pi}{4}+k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{3\pi}{4},\frac{7\pi}{4}\right\}$.

34. $\sin\left(3\theta+\frac{\pi}{18}\right)=1$

$3\theta+\frac{\pi}{18}=\frac{\pi}{2}+2k\pi$

$3\theta=\frac{4\pi}{9}+2k\pi$

$\theta=\frac{4\pi}{27}+\frac{2k\pi}{3}$, k is any integer

On the interval $0\le\theta<2\pi$, the solution set is $\left\{\frac{4\pi}{27},\frac{22\pi}{27},\frac{40\pi}{27}\right\}$.

35. $\tan\left(\frac{\theta}{2}+\frac{\pi}{3}\right)=1$

$\frac{\theta}{2}+\frac{\pi}{3}=\frac{\pi}{4}+k\pi$

$\frac{\theta}{2}=-\frac{\pi}{12}+k\pi$

$\theta=-\frac{\pi}{6}+2k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{11\pi}{6}\right\}$.

36. $\cos\left(\frac{\theta}{3}-\frac{\pi}{4}\right)=\frac{1}{2}$

$\frac{\theta}{3}-\frac{\pi}{4}=\frac{\pi}{3}+2k\pi$ or $\frac{\theta}{3}-\frac{\pi}{4}=\frac{5\pi}{3}+2k\pi$

$\frac{\theta}{3}=\frac{7\pi}{12}+2k\pi$ or $\frac{\theta}{3}=\frac{23\pi}{12}+2k\pi$

$\theta=\frac{7\pi}{4}+6k\pi$ or $\theta=\frac{23\pi}{4}+6k\pi$,

k is any integer.

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{7\pi}{4}\right\}$.

37. $\sin\theta=\frac{1}{2}$

$\left\{\theta\,\middle|\,\theta=\frac{\pi}{6}+2k\pi \text{ or } \theta=\frac{5\pi}{6}+2k\pi\right\}$, k is any integer. Six solutions are

$\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6},\frac{25\pi}{6},\frac{29\pi}{6}$.

38. $\tan\theta=1$

$\left\{\theta\,\middle|\,\theta=\frac{\pi}{4}+k\pi\right\}$, k is any integer

Six solutions are $\theta=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\frac{13\pi}{4},\frac{17\pi}{4},\frac{21\pi}{4}$.

39. $\tan\theta=-\frac{\sqrt{3}}{3}$

$\left\{\theta\,\middle|\,\theta=\frac{5\pi}{6}+k\pi\right\}$, k is any integer

Six solutions are

$\theta=\frac{5\pi}{6},\frac{11\pi}{6},\frac{17\pi}{6},\frac{23\pi}{6},\frac{29\pi}{6},\frac{35\pi}{6}$.

40. $\cos\theta=-\frac{\sqrt{3}}{2}$

$\left\{\theta\,\middle|\,\theta=\frac{5\pi}{6}+2k\pi \text{ or } \theta=\frac{7\pi}{6}+2k\pi\right\}$, k is any integer. Six solutions are

$\theta=\frac{5\pi}{6},\frac{7\pi}{6},\frac{17\pi}{6},\frac{19\pi}{6},\frac{29\pi}{6},\frac{31\pi}{6}$.

41. $\cos\theta=0$

$\left\{\theta\,\middle|\,\theta=\frac{\pi}{2}+2k\pi \text{ or } \theta=\frac{3\pi}{2}+2k\pi\right\}$, k is any integer

Six solutions are $\theta=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\frac{9\pi}{2},\frac{11\pi}{2}$.

42. $\sin\theta=\frac{\sqrt{2}}{2}$

$\left\{\theta\,\middle|\,\theta=\frac{\pi}{4}+2k\pi \text{ or } \theta=\frac{3\pi}{4}+2k\pi\right\}$, k is any integer

Six solutions are $\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{9\pi}{4},\frac{11\pi}{4},\frac{17\pi}{4},\frac{19\pi}{4}$.

43. $\cos(2\theta) = -\frac{1}{2}$

$2\theta = \frac{2\pi}{3} + 2k\pi$ or $2\theta = \frac{4\pi}{3} + 2k\pi$, k is any integer

$\left\{\theta \middle| \theta = \frac{\pi}{3} + k\pi \text{ or } \theta = \frac{2\pi}{3} + k\pi\right\}$, k is any integer

Six solutions are $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$.

44. $\sin(2\theta) = -1$

$2\theta = \frac{3\pi}{2} + 2k\pi$, k is any integer

$\left\{\theta \middle| \theta = \frac{3\pi}{4} + k\pi\right\}$, k is any integer

Six solutions are

$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$.

45. $\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{2}$

$\frac{\theta}{2} = \frac{4\pi}{3} + 2k\pi$ or $\frac{\theta}{2} = \frac{5\pi}{3} + 2k\pi$, k is any integer

$\left\{\theta \middle| \theta = \frac{8\pi}{3} + 4k\pi \text{ or } \theta = \frac{10\pi}{3} + 4k\pi\right\}$, k is any integer. Six solutions are

$\theta = \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3}, \frac{32\pi}{3}, \frac{34\pi}{3}$.

46. $\tan\frac{\theta}{2} = -1$

$\frac{\theta}{2} = \frac{3\pi}{4} + k\pi$, k is any integer

$\left\{\theta \middle| \theta = \frac{3\pi}{2} + 2k\pi\right\}$, k is any integer

Six solutions are

$\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}, \frac{19\pi}{2}, \frac{23\pi}{2}$.

47. $\sin\theta = 0.4$

$\theta = \sin^{-1}(0.4) \approx 0.41$

$\theta \approx 0.41$ or $\theta \approx \pi - 0.41 \approx 2.73$.

The solution set is $\{0.41, 2.73\}$.

48. $\cos\theta = 0.6$

$\theta = \cos^{-1}(0.6) \approx 0.93$

$\theta \approx 0.93$ or $\theta \approx 2\pi - 0.93 \approx 5.36$.

The solution set is $\{0.93, 5.36\}$.

49. $\tan\theta = 5$

$\theta = \tan^{-1}(5) \approx 1.37$

$\theta \approx 1.37$ or $\theta \approx \pi + 1.37 \approx 4.51$.

The solution set is $\{1.37, 4.51\}$.

50. $\cot\theta = 2$

$\tan\theta = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.46$

$\theta \approx 0.46$ or $\theta \approx \pi + 0.46 \approx 3.61$.

The solution set is $\{0.46, 3.61\}$.

51. $\cos\theta = -0.9$

$\theta = \cos^{-1}(-0.9) \approx 2.69$

$\theta \approx 2.69$ or $\theta \approx 2\pi - 2.69 \approx 3.59$.

The solution set is $\{2.69, 3.59\}$.

52. $\sin\theta = -0.2$

$\theta = \sin^{-1}(-0.2) \approx -0.20$

$\theta \approx -0.20 + 2\pi$ or $\theta \approx \pi - (-0.20)$.

≈ 6.08 $\qquad\qquad \approx 3.34$

The solution set is $\{3.34, 6.08\}$.

53. $\sec\theta = -4$

$\cos\theta = -\frac{1}{4}$

$\theta = \cos^{-1}\left(-\frac{1}{4}\right) \approx 1.82$

$\theta \approx 1.82$ or $\theta \approx 2\pi - 1.82 \approx 4.46$.

The solution set is $\{1.82, 4.46\}$.

54. $\csc\theta = -3$

$\sin\theta = -\frac{1}{3}$

$\theta = \sin^{-1}\left(-\frac{1}{3}\right) \approx -0.34$

$\theta \approx -0.34 + 2\pi$ or $\theta \approx \pi - (-0.34)$.

≈ 5.94 $\qquad\qquad \approx 3.48$

The solution set is $\{3.48, 5.94\}$.

55. $5\tan\theta + 9 = 0$

$5\tan\theta = -9$

$\tan\theta = -\frac{9}{5}$

$\theta = \tan^{-1}\left(-\frac{9}{5}\right) \approx -1.064$

$\theta \approx -1.064 + \pi$ or $\theta \approx -1.064 + 2\pi$

≈ 2.08 $\qquad \approx 5.22$

The solution set is $\{2.08,\ 5.22\}$.

56. $4\cot\theta = -5$

$\cot\theta = -\frac{5}{4}$

$\tan\theta = -\frac{4}{5}$

$\theta = \tan^{-1}\left(-\frac{4}{5}\right) \approx -0.675$

$\theta \approx -0.675 + \pi$ or $\theta \approx -0.675 + 2\pi$.

≈ 2.47 $\qquad \approx 5.61$

The solution set is $\{2.47,\ 5.61\}$.

57. $3\sin\theta - 2 = 0$

$3\sin\theta = 2$

$\sin\theta = \frac{2}{3}$

$\theta = \sin^{-1}\left(\frac{2}{3}\right) \approx 0.73$

$\theta \approx 0.73$ or $\theta \approx \pi - 0.73 \approx 2.41$.

The solution set is $\{0.73,\ 2.41\}$.

58. $4\cos\theta + 3 = 0$

$4\cos\theta = -3$

$\cos\theta = -\frac{3}{4}$

$\theta = \cos^{-1}\left(-\frac{3}{4}\right) \approx 2.42$

$\theta \approx 2.42$ or $\theta \approx 2\pi - 2.42 \approx 3.86$.

The solution set is $\{2.42,\ 3.86\}$.

59. $2\cos^2\theta + \cos\theta = 0$

$\cos\theta(2\cos\theta + 1) = 0$

$\cos\theta = 0$ or $2\cos\theta + 1 = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad 2\cos\theta = -1$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

The solution set is $\left\{\frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{3\pi}{2}\right\}$.

60. $\sin^2\theta - 1 = 0$

$(\sin\theta + 1)(\sin\theta - 1) = 0$

$\sin\theta + 1 = 0$ or $\sin\theta - 1 = 0$

$\sin\theta = -1$ $\qquad \sin\theta = 1$

$\theta = \frac{3\pi}{2}$ $\qquad \theta = \frac{\pi}{2}$

The solution set is $\left\{\frac{\pi}{2},\ \frac{3\pi}{2}\right\}$.

61. $2\sin^2\theta - \sin\theta - 1 = 0$

$(2\sin\theta + 1)(\sin\theta - 1) = 0$

$2\sin\theta + 1 = 0$ or $\sin\theta - 1 = 0$

$2\sin\theta = -1$ $\qquad \sin\theta = 1$

$\sin\theta = -\frac{1}{2}$ $\qquad \theta = \frac{\pi}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

The solution set is $\left\{\frac{\pi}{2},\ \frac{7\pi}{6},\ \frac{11\pi}{6}\right\}$.

62. $2\cos^2\theta + \cos\theta - 1 = 0$

$(\cos\theta + 1)(2\cos\theta - 1) = 0$

$\cos\theta + 1 = 0$ or $2\cos\theta - 1 = 0$

$\cos\theta = -1$ $\qquad 2\cos\theta = 1$

$\theta = \pi$ $\qquad \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

The solution set is $\left\{\frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}\right\}$.

63. $(\tan\theta-1)(\sec\theta-1)=0$

$\tan\theta-1=0$ or $\sec\theta-1=0$

$\tan\theta=1 \qquad \sec\theta=1$

$\theta=\frac{\pi}{4},\frac{5\pi}{4} \qquad \theta=0$

The solution set is $\left\{0,\ \frac{\pi}{4},\ \frac{5\pi}{4}\right\}$.

64. $(\cot\theta+1)\left(\csc\theta-\frac{1}{2}\right)=0$

$\cot\theta+1=0$ or $\csc\theta-\frac{1}{2}=0$

$\cot\theta=-1 \qquad \csc\theta=\frac{1}{2}$

$\theta=\frac{3\pi}{4},\frac{7\pi}{4}$ (not possible)

The solution set is $\left\{\frac{3\pi}{4},\ \frac{7\pi}{4}\right\}$.

65.
$$\sin^2\theta-\cos^2\theta=1+\cos\theta$$
$$\left(1-\cos^2\theta\right)-\cos^2\theta=1+\cos\theta$$
$$1-2\cos^2\theta=1+\cos\theta$$
$$2\cos^2\theta+\cos\theta=0$$
$$(\cos\theta)(2\cos\theta+1)=0$$

$\cos\theta=0$ or $2\cos\theta+1=0$

$\theta=\frac{\pi}{2},\frac{3\pi}{2} \qquad \cos\theta=-\frac{1}{2}$

$\theta=\frac{2\pi}{3},\frac{4\pi}{3}$

The solution set is $\left\{\frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{3\pi}{2}\right\}$.

66.
$$\cos^2\theta-\sin^2\theta+\sin\theta=0$$
$$\left(1-\sin^2\theta\right)-\sin^2\theta+\sin\theta=0$$
$$1-2\sin^2\theta+\sin\theta=0$$
$$2\sin^2\theta-\sin\theta-1=0$$
$$(2\sin\theta+1)(\sin\theta-1)=0$$

$2\sin\theta+1=0$ or $\sin\theta-1=0$

$\sin\theta=-\frac{1}{2} \qquad \sin\theta=1$

$\theta=\frac{7\pi}{6},\frac{11\pi}{6} \qquad \theta=\frac{\pi}{2}$

The solution set is $\left\{\frac{\pi}{2},\ \frac{7\pi}{6},\ \frac{11\pi}{6}\right\}$.

67.
$$\sin^2\theta=6\left(\cos(-\theta)+1\right)$$
$$\sin^2\theta=6\left(\cos(\theta)+1\right)$$
$$1-\cos^2\theta=6\cos\theta+6$$
$$\cos^2\theta+6\cos\theta+5=0$$
$$(\cos\theta+5)(\cos\theta+1)=0$$

$\cos\theta+5=0$ or $\cos\theta+1=0$

$\cos\theta=-5 \qquad \cos\theta=-1$

(not possible) $\qquad \theta=\pi$

The solution set is $\{\pi\}$.

68.
$$2\sin^2\theta=3\left(1-\cos(-\theta)\right)$$
$$2\sin^2\theta=3(1-\cos\theta)$$
$$2\left(1-\cos^2\theta\right)=3(1-\cos\theta)$$
$$2-2\cos^2\theta=3-3\cos\theta$$
$$2\cos^2\theta-3\cos\theta+1=0$$
$$(2\cos\theta-1)(\cos\theta-1)=0$$

$2\cos\theta-1=0$ or $\cos\theta-1=0$

$\cos\theta=\frac{1}{2} \qquad \cos\theta=1$

$\theta=0$

$\theta=\frac{\pi}{3},\frac{5\pi}{3}$

The solution set is $\left\{0,\ \frac{\pi}{3},\ \frac{5\pi}{3}\right\}$.

69.
$$\cos\theta=-\sin(-\theta)$$
$$\cos\theta=-(-\sin\theta)$$
$$\cos\theta=\sin\theta$$
$$\frac{\sin\theta}{\cos\theta}=1$$
$$\tan\theta=1$$
$$\theta=\frac{\pi}{4},\frac{5\pi}{4}$$

The solution set is $\left\{\frac{\pi}{4},\ \frac{5\pi}{4}\right\}$.

70. $\cos\theta - \sin(-\theta) = 0$

$\cos\theta - (-\sin(\theta)) = 0$

$\cos\theta + \sin\theta = 0$

$\sin\theta = -\cos\theta$

$\frac{\sin\theta}{\cos\theta} = -1$

$\tan\theta = -1$

$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

The solution set is $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$.

71. $\tan\theta = 2\sin\theta$

$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$

$\sin\theta = 2\sin\theta\cos\theta$

$0 = 2\sin\theta\cos\theta - \sin\theta$

$0 = \sin\theta(2\cos\theta - 1)$

$2\cos\theta - 1 = 0$ or $\sin\theta = 0$

$\cos\theta = \frac{1}{2}$ $\quad\theta = 0, \pi$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

The solution set is $\left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$.

72. $\tan\theta = \cot\theta$

$\tan\theta = \frac{1}{\tan\theta}$

$\tan^2\theta = 1$

$\tan\theta = \pm 1$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

The solution set is $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$.

73. $1 + \sin\theta = 2\cos^2\theta$

$1 + \sin\theta = 2(1 - \sin^2\theta)$

$1 + \sin\theta = 2 - 2\sin^2\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$

$\sin\theta = \frac{1}{2}$ $\quad\sin\theta = -1$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad\theta = \frac{3\pi}{2}$

The solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$.

74. $\sin^2\theta = 2\cos\theta + 2$

$1 - \cos^2\theta = 2\cos\theta + 2$

$\cos^2\theta + 2\cos\theta + 1 = 0$

$(\cos\theta + 1)^2 = 0$

$\cos\theta + 1 = 0$

$\cos\theta = -1$

$\theta = \pi$

The solution set is $\{\pi\}$.

75. $2\sin^2\theta - 5\sin\theta + 3 = 0$

$(2\sin\theta - 3)(\sin\theta + 1) = 0$

$2\sin\theta - 3 = 0$ or $\sin\theta - 1 = 0$

$\sin\theta = \frac{3}{2}$ (not possible) $\quad\theta = \frac{\pi}{2}$

The solution set is $\left\{\frac{\pi}{2}\right\}$.

76. $2\cos^2\theta - 7\cos\theta - 4 = 0$

$(2\cos\theta + 1)(\cos\theta - 4) = 0$

$2\cos\theta + 1 = 0$ or $\cos\theta - 4 = 0$

$\sin\theta = -\frac{1}{2}$ $\quad\cos\theta = 4$ (not possible)

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

The solution set is $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

77.
$$3(1-\cos\theta)=\sin^2\theta$$
$$3-3\cos\theta=1-\cos^2\theta$$
$$\cos^2\theta-3\cos\theta+2=0$$
$$(\cos\theta-1)(\cos\theta-2)=0$$
$$\cos\theta-1=0 \quad \text{or} \quad \cos\theta-2=0$$
$$\cos\theta=1 \qquad \cos\theta=2$$
$$\theta=0 \qquad \text{(not possible)}$$
The solution set is $\{0\}$.

78.
$$4(1+\sin\theta)=\cos^2\theta$$
$$4+4\sin\theta=1-\sin^2\theta$$
$$\sin^2\theta+4\sin\theta+3=0$$
$$(\sin\theta+1)(\sin\theta+3)=0$$
$$\sin\theta+1=0 \quad \text{or} \quad \sin\theta+3=0$$
$$\sin\theta=-1 \qquad \sin\theta=-3$$
$$\theta=\frac{3\pi}{2} \qquad \text{(not possible)}$$
The solution set is $\left\{\frac{3\pi}{2}\right\}$.

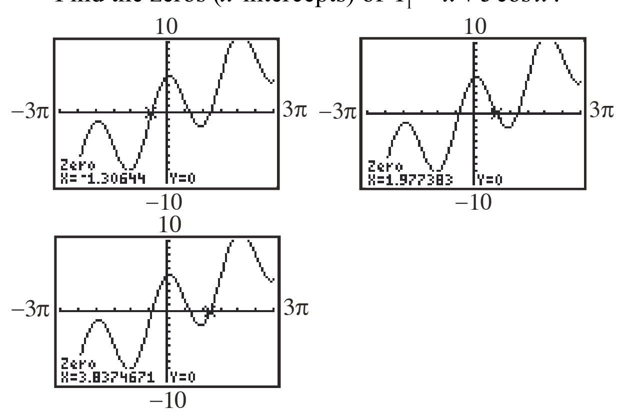
79.
$$\tan^2\theta=\frac{3}{2}\sec\theta$$
$$\sec^2\theta-1=\frac{3}{2}\sec\theta$$
$$2\sec^2\theta-2=3\sec\theta$$
$$2\sec^2\theta-3\sec\theta-2=0$$
$$(2\sec\theta+1)(\sec\theta-2)=0$$
$$2\sec\theta+1=0 \quad \text{or} \quad \sec\theta-2=0$$
$$\sec\theta=-\frac{1}{2} \qquad \sec\theta=2$$
$$\text{(not possible)} \qquad \theta=\frac{\pi}{3},\frac{5\pi}{3}$$
The solution set is $\left\{\frac{\pi}{3},\ \frac{5\pi}{3}\right\}$.

80.
$$\csc^2\theta=\cot\theta+1$$
$$1+\cot^2\theta=\cot\theta+1$$
$$\cot^2\theta-\cot\theta=0$$
$$\cot\theta(\cot\theta-1)=0$$
$$\cot\theta=0 \quad \text{or} \quad \cot\theta=1$$
$$\theta=\frac{\pi}{2},\frac{3\pi}{2} \qquad \theta=\frac{\pi}{4},\frac{5\pi}{4}$$
The solution set is $\left\{\frac{\pi}{4},\ \frac{\pi}{2},\ \frac{5\pi}{4},\ \frac{3\pi}{2}\right\}$.

81.
$$\sec^2\theta+\tan\theta=0$$
$$\tan^2\theta+1+\tan\theta=0$$
This equation is quadratic in $\tan\theta$.
The discriminant is $b^2-4ac=1-4=-3<0$.
The equation has no real solutions.

82.
$$\sec\theta=\tan\theta+\cot\theta$$
$$\frac{1}{\cos\theta}=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$$
$$\frac{1}{\cos\theta}=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$
$$\frac{1}{\cos\theta}=\frac{1}{\sin\theta\cos\theta}$$
$$\frac{\sin\theta\cos\theta}{\cos\theta}=1$$
$$\sin\theta=1$$
$$\theta=\frac{\pi}{2}$$
Since $\sec\left(\frac{\pi}{2}\right)$ and $\tan\left(\frac{\pi}{2}\right)$ do not exist, the equation has no real solutions.

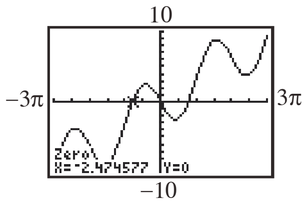
83. $x+5\cos x=0$
Find the zeros (x-intercepts) of $Y_1=x+5\cos x$:

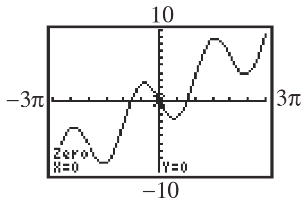


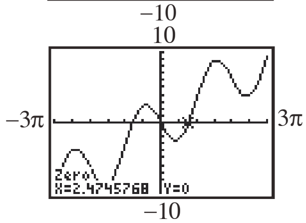
$x\approx-1.31, 1.98, 3.84$

84. $x - 4\sin x = 0$

Find the zeros (x-intercepts) of $Y_1 = x - 4\sin x$:



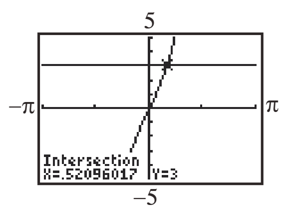




$x \approx -2.47, 0, 2.47$

85. $22x - 17\sin x = 3$

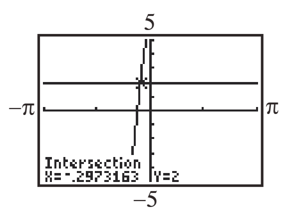
Find the intersection of $Y_1 = 22x - 17\sin x$ and $Y_2 = 3$:



$x \approx 0.52$

86. $19x + 8\cos x = 2$

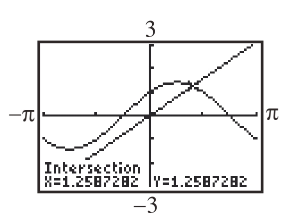
Find the intersection of $Y_1 = 19x + 8\cos x$ and $Y_2 = 2$:



$x \approx -0.30$

87. $\sin x + \cos x = x$

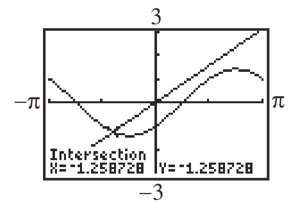
Find the intersection of $Y_1 = \sin x + \cos x$ and $Y_2 = x$:



$x \approx 1.26$

88. $\sin x - \cos x = x$

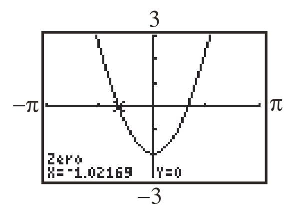
Find the intersection of $Y_1 = \sin x - \cos x$ and $Y_2 = x$:

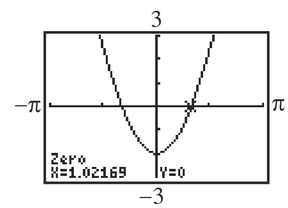


$x \approx -1.26$

89. $x^2 - 2\cos x = 0$

Find the zeros (x-intercepts) of $Y_1 = x^2 - 2\cos x$:

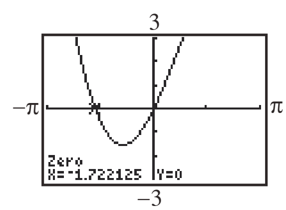


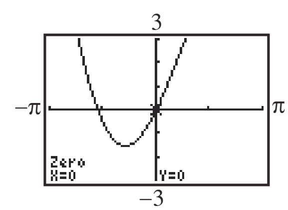


$x \approx -1.02, 1.02$

90. $x^2 + 3\sin x = 0$

Find the zeros (x-intercepts) of $Y_1 = x^2 + 3\sin x$:

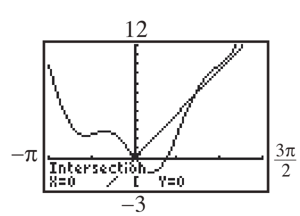


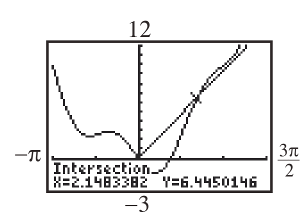


$x \approx -1.72, 0$

91. $x^2 - 2\sin(2x) = 3x$

Find the intersection of $Y_1 = x^2 - 2\sin(2x)$ and $Y_2 = 3x$:

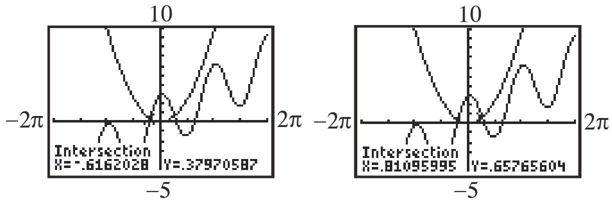




$x \approx 0, 2.15$

92. $x^2 = x + 3\cos(2x)$

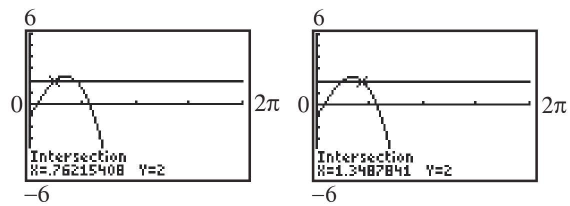
Find the intersection of $Y_1 = x^2$ and $Y_2 = x + 3\cos(2x)$:



$x \approx -0.62, 0.81$

93. $6\sin x - e^x = 2,\ x > 0$

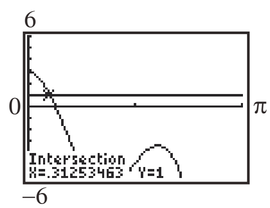
Find the intersection of $Y_1 = 6\sin x - e^x$ and $Y_2 = 2$:



$x \approx 0.76, 1.35$

94. $4\cos(3x) - e^x = 1,\ x > 0$

Find the intersection of $Y_1 = 4\cos(3x) - e^x$ and $Y_2 = 1$:



$x \approx 0.31$

95.
$$f(x) = 0$$
$$4\sin^2 x - 3 = 0$$
$$4\sin^2 x = 3$$
$$\sin^2 x = \frac{3}{4}$$
$$\sin x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3} + k\pi \text{ or } x = \frac{2\pi}{3} + k\pi \text{, } k \text{ is any integer}$$

On the interval $[0, 2\pi]$, the zeros of f are $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

96.
$$f(x) = 0$$
$$2\cos(3x) + 1 = 0$$
$$2\cos(3x) = -1$$
$$\cos(3x) = -\frac{1}{2}$$
$$3x = \frac{2\pi}{3} + 2k\pi \text{ or } 3x = \frac{4\pi}{3} + 2k\pi$$
$$x = \frac{2\pi}{9} + \frac{2k\pi}{3} \text{ or } x = \frac{4\pi}{9} + \frac{2k\pi}{3},$$

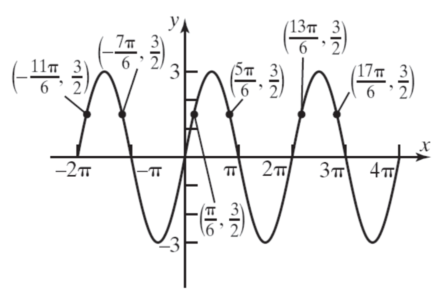
k is any integer

On the interval $[0, \pi]$, the zeros of f are $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$.

97. a.
$$f(x) = 0$$
$$3\sin x = 0$$
$$\sin x = 0$$
$$x = 0 + 2k\pi \text{ or } x = \pi + 2k\pi,\ k \text{ is any integer}$$

On the interval $[-2\pi, 4\pi]$, the zeros of f are $-2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi$.

b. $f(x) = 3\sin x$



c.
$$f(x) = \frac{3}{2}$$
$$3\sin x = \frac{3}{2}$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6} + 2k\pi \text{ or } x = \frac{5\pi}{6} + 2k\pi,\ k \text{ is any integer}$$

On the interval $[-2\pi, 4\pi]$, the solution set is $\left\{-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}\right\}$.

d. From the graph in part (b) and the results of part (c), the solutions of $f(x) > \frac{3}{2}$ on the interval $[-2\pi, 4\pi]$ is $\left\{x \middle| -\frac{11\pi}{6} < x < -\frac{7\pi}{6}\right.$ or $\frac{\pi}{6} < x < \frac{5\pi}{6}$ or $\left.\frac{13\pi}{6} < x < \frac{17\pi}{6}\right\}$.

98. a. $f(x) = 0$

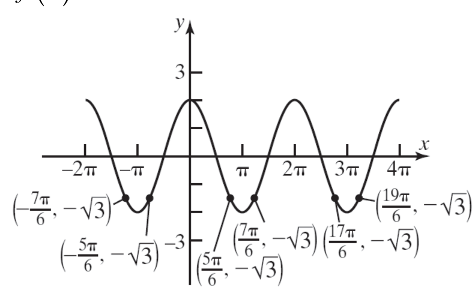
$2\cos x = 0$

$\cos x = 0$

$x = \frac{\pi}{2} + 2k\pi$ or $x = \frac{3\pi}{2} + 2k\pi$, k is any integer

On the interval $[-2\pi, 4\pi]$, the zeros of f are $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

b. $f(x) = 2\cos x$



c. $f(x) = -\sqrt{3}$

$2\cos x = -\sqrt{3}$

$\cos x = -\frac{\sqrt{3}}{2}$

$x = \frac{5\pi}{6} + 2k\pi$ or $x = \frac{7\pi}{6} + 2k\pi$, k is any integer

On the interval $[-2\pi, 4\pi]$, the solution set is $\left\{-\frac{7\pi}{6}, -\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}\right\}$.

d. From the graph in part (b) and the results of part (c), the solutions of $f(x) < -\sqrt{3}$ on the interval $[-2\pi, 4\pi]$ is $\left\{x \middle| -\frac{7\pi}{6} < x < -\frac{5\pi}{6}\right.$ or $\frac{5\pi}{6} < x < \frac{7\pi}{6}$ or $\left.\frac{17\pi}{6} < x < \frac{19\pi}{6}\right\}$.

99. $f(x) = 4\tan x$

a. $f(x) = -4$

$4\tan x = -4$

$\tan x = -1$

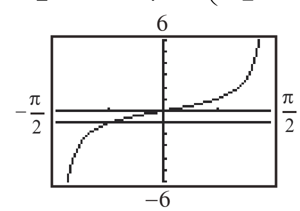
$\left\{x \middle| x = -\frac{\pi}{4} + k\pi\right\}$, k is any integer

b. $f(x) < -4$

$4\tan x < -4$

$\tan x < -1$

Graphing $y_1 = \tan x$ and $y_2 = -1$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we see that $y_1 < y_2$ for $-\frac{\pi}{2} < x < -\frac{\pi}{4}$ or $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$.



100. $f(x) = \cot x$

a. $f(x) = -\sqrt{3}$

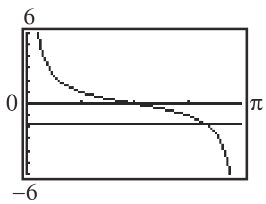
$\cot x = -\sqrt{3}$

$\left\{x \middle| x = \frac{5\pi}{6} + k\pi\right\}$, k is any integer

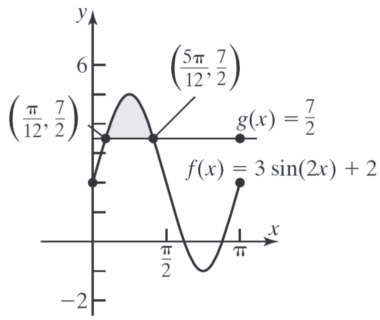
b. $f(x) > -\sqrt{3}$

$\cot x > -\sqrt{3}$

Graphing $y_1 = \frac{1}{\tan x}$ and $y_2 = -\sqrt{3}$ on the interval $(0, \pi)$, we see that $y_1 > y_2$ for $0 < x < \frac{5\pi}{6}$ or $\left(0, \frac{5\pi}{6}\right)$.



101. a, d. $f(x) = 3\sin(2x) + 2$; $g(x) = \frac{7}{2}$



b.

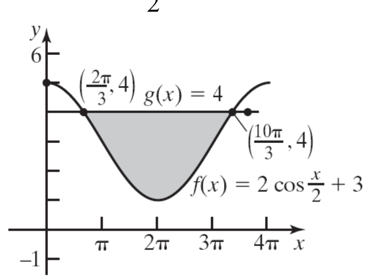
$$f(x) = g(x)$$
$$3\sin(2x) + 2 = \frac{7}{2}$$
$$3\sin(2x) = \frac{3}{2}$$
$$\sin(2x) = \frac{1}{2}$$
$$2x = \frac{\pi}{6} + 2k\pi \text{ or } 2x = \frac{5\pi}{6} + 2k\pi$$
$$x = \frac{\pi}{12} + k\pi \text{ or } x = \frac{5\pi}{12} + k\pi,$$

k is any integer

On $[0, \pi]$, the solution set is $\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}$.

c. From the graph in part (a) and the results of part (b), the solution of $f(x) > g(x)$ on $[0, \pi]$ is $\left\{x \middle| \frac{\pi}{12} < x < \frac{5\pi}{12}\right\}$ or $\left(\frac{\pi}{12}, \frac{5\pi}{12}\right)$.

102. a, d. $f(x) = 2\cos\frac{x}{2} + 3$; $g(x) = 4$



b.

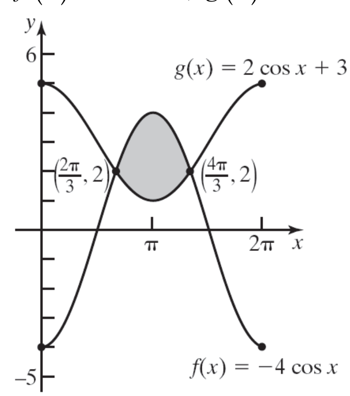
$$f(x) = g(x)$$
$$2\cos\frac{x}{2} + 3 = 4$$
$$2\cos\frac{x}{2} = 1$$
$$\cos\frac{x}{2} = \frac{1}{2}$$
$$\frac{x}{2} = \frac{\pi}{3} + 2k\pi \text{ or } \frac{x}{2} = \frac{5\pi}{3} + 2k\pi$$
$$x = \frac{2\pi}{3} + 4k\pi \text{ or } x = \frac{10\pi}{3} + 4k\pi,$$

k is any integer

On $[0, 4\pi]$, the solution set is $\left\{\frac{2\pi}{3}, \frac{10\pi}{3}\right\}$.

c. From the graph in part (a) and the results of part (b), the solution of $f(x) < g(x)$ on $[0, 4\pi]$ is $\left\{x \middle| \frac{2\pi}{3} < x < \frac{10\pi}{3}\right\}$ or $\left(\frac{2\pi}{3}, \frac{10\pi}{3}\right)$.

103. a, d. $f(x) = -4\cos x$; $g(x) = 2\cos x + 3$



b.

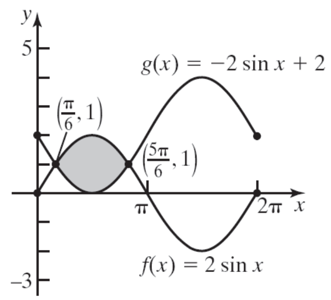
$$f(x) = g(x)$$
$$-4\cos x = 2\cos x + 3$$
$$-6\cos x = 3$$
$$\cos x = \frac{3}{-6} = -\frac{1}{2}$$
$$x = \frac{2\pi}{3} + 2k\pi \text{ or } x = \frac{4\pi}{3} + 2k\pi,$$

k is any integer

On $[0, 2\pi]$, the solution set is $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

c. From the graph in part (a) and the results of part (b), the solution of $f(x) > g(x)$ on $[0, 2\pi]$ is $\left\{x \middle| \frac{2\pi}{3} < x < \frac{4\pi}{3}\right\}$ or $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$.

104. a, d. $f(x) = 2\sin x$; $g(x) = -2\sin x + 2$



b. $f(x) = g(x)$

$2\sin x = -2\sin x + 2$

$4\sin x = 2$

$\sin x = \frac{2}{4} = \frac{1}{2}$

$x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$,

k is any integer

On $[0, 2\pi]$, the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

c. From the graph in part (a) and the results of part (b), the solution of $f(x) > g(x)$ on $[0, 2\pi]$ is $\left\{x \middle| \frac{\pi}{6} < x < \frac{5\pi}{6}\right\}$ or $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$.

105. $P(t) = 100 + 20\sin\left(\frac{7\pi}{3}t\right)$

a. Solve $P(t) = 100$ on the interval $[0, 1]$.

$100 + 20\sin\left(\frac{7\pi}{3}t\right) = 100$

$20\sin\left(\frac{7\pi}{3}t\right) = 0$

$\sin\left(\frac{7\pi}{3}t\right) = 0$

$\frac{7\pi}{3}t = k\pi$, k is any integer

$t = \frac{3}{7}k$, k is any integer

We need $0 \le \frac{3}{7}k \le 1$, or $0 \le k \le \frac{7}{3}$.

For $k = 0$, $t = 0$ sec.

For $k = 1$, $t = \frac{3}{7} \approx 0.43$ sec.

For $k = 2$, $t = \frac{6}{7} \approx 0.86$ sec.

The blood pressure will be 100 mmHg after 0 seconds, 0.43 seconds, and 0.86 seconds.

b. Solve $P(t) = 120$ on the interval $[0, 1]$.

$100 + 20\sin\left(\frac{7\pi}{3}t\right) = 120$

$20\sin\left(\frac{7\pi}{3}t\right) = 20$

$\sin\left(\frac{7\pi}{3}t\right) = 1$

$\frac{7\pi}{3}t = 2\pi k + \frac{\pi}{2}$, k is any integer

$t = \frac{3\left(2k + \frac{1}{2}\right)}{7}$, k is any integer

We need

$0 \le \frac{3\left(2k + \frac{1}{2}\right)}{7} \le 1$

$0 \le 2k + \frac{1}{2} \le \frac{7}{3}$

$-\frac{1}{2} \le 2k \le \frac{11}{6}$

$-\frac{1}{4} \le k \le \frac{11}{12}$

For $k = 0$, $t = \frac{3}{14} \approx 0.21$ sec

The blood pressure will be 120mmHg after 0.21 sec.

c. Solve $P(t) = 105$ on the interval $[0, 1]$.

$100 + 20\sin\left(\frac{7\pi}{3}t\right) = 105$

$20\sin\left(\frac{7\pi}{3}t\right) = 5$

$\sin\left(\frac{7\pi}{3}t\right) = \frac{3}{4}$

$\frac{7\pi}{3}t = \sin^{-1}\left(\frac{3}{4}\right)$

$t = \frac{3}{7\pi}\sin^{-1}\left(\frac{3}{4}\right)$

On the interval $[0, 1]$, we get $t \approx 0.03$ seconds, $t \approx 0.39$ seconds, and $t \approx 0.89$ seconds. Using this information, along with the results from part (a), the blood pressure will be between 100 mmHg and 105 mmHg for values of t (in seconds) in the interval $[0,0.03]\cup[0.39,0.43]\cup[0.86,0.89]$.

106. $h(t)=125\sin\left(0.157t-\frac{\pi}{2}\right)+125$

a. Solve $h(t)=125\sin\left(0.157t-\frac{\pi}{2}\right)+125=125$ on the interval $[0,40]$.

$$125\sin\left(0.157t-\frac{\pi}{2}\right)+125=125$$
$$125\sin\left(0.157t-\frac{\pi}{2}\right)=0$$
$$\sin\left(0.157t-\frac{\pi}{2}\right)=0$$
$$0.157t-\frac{\pi}{2}=k\pi,\ k \text{ is any integer}$$
$$0.157t=k\pi+\frac{\pi}{2},\ k \text{ is any integer}$$
$$t=\frac{k\pi+\frac{\pi}{2}}{0.157},\ k \text{ is any integer}$$

For $k=0$, $t=\frac{0+\frac{\pi}{2}}{0.157}\approx 10$ seconds.

For $k=1$, $t=\frac{\pi+\frac{\pi}{2}}{0.157}\approx 30$ seconds.

For $k=2$, $t=\frac{2\pi+\frac{\pi}{2}}{0.157}\approx 50$ seconds.

So during the first 40 seconds, an individual on the Ferris wheel is exactly 125 feet above the ground when $t\approx 10$ seconds and again when $t\approx 30$ seconds.

b. Solve $h(t)=125\sin\left(0.157t-\frac{\pi}{2}\right)+125=250$ on the interval $[0,80]$.

$$125\sin\left(0.157t-\frac{\pi}{2}\right)+125=250$$
$$125\sin\left(0.157t-\frac{\pi}{2}\right)=125$$
$$\sin\left(0.157t-\frac{\pi}{2}\right)=1$$
$$0.157t-\frac{\pi}{2}=\frac{\pi}{2}+2k\pi,\ k \text{ is any integer}$$
$$0.157t=\pi+2k\pi,\ k \text{ is any integer}$$
$$t=\frac{\pi+2k\pi}{0.157},\ k \text{ is any integer}$$

For $k=0$, $t=\frac{\pi}{0.157}\approx 20$ seconds.

For $k=1$, $t=\frac{\pi+2\pi}{0.157}\approx 60$ seconds.

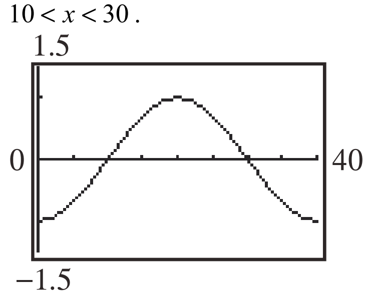
For $k=2$, $t=\frac{\pi+4\pi}{0.157}\approx 100$ seconds.

So during the first 80 seconds, an individual on the Ferris wheel is exactly 250 feet above the ground when $t\approx 20$ seconds and again when $t\approx 60$ seconds.

c. Solve $h(t)=125\sin\left(0.157t-\frac{\pi}{2}\right)+125>125$ on the interval $[0,40]$.

$$125\sin\left(0.157t-\frac{\pi}{2}\right)+125>125$$
$$125\sin\left(0.157t-\frac{\pi}{2}\right)>0$$
$$\sin\left(0.157t-\frac{\pi}{2}\right)>0$$

Graphing $y_1=\sin\left(0.157x-\frac{\pi}{2}\right)$ and $y_2=0$ on the interval $[0,40]$, we see that $y_1>y_2$ for $10<x<30$.



So during the first 40 seconds, an individual on the Ferris wheel is more than 125 feet above the ground for times between about 10 and 30 seconds. That is, on the interval $10<x<30$, or $(10,30)$.

107. $d(x) = 70\sin(0.65x) + 150$

a. $d(0) = 70\sin(0.65(0)) + 150$
$= 70\sin(0) + 150$
$= 150$ miles

b. Solve $d(x) = 70\sin(0.65x) + 150 = 100$ on the interval $[0, 20]$.

$$70\sin(0.65x) + 150 = 100$$
$$70\sin(0.65x) = -50$$
$$\sin(0.65x) = -\frac{5}{7}$$
$$0.65x = \sin^{-1}\left(-\frac{5}{7}\right) + 2\pi k$$
$$x = \frac{\sin^{-1}\left(-\frac{5}{7}\right) + 2\pi k}{0.65}$$
$$x \approx \frac{3.94 + 2\pi k}{0.65} \text{ or } x \approx \frac{5.94 + 2\pi k}{0.65},$$

k is any integer

For $k = 0$, $x \approx \frac{3.94 + 0}{0.65} \approx 6.06$ min or $x \approx \frac{5.94 + 0}{0.65} \approx 8.44$ min

For $k = 1$, $x \approx \frac{3.94 + 2\pi}{0.65} \approx 15.72$ min or $x \approx \frac{5.94 + 2\pi}{0.65} \approx 18.11$ min

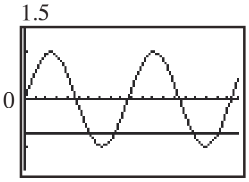
For $k = 2$,
$x \approx \frac{3.94 + 4\pi}{0.65} \approx 25.39$ min or $x \approx \frac{5.94 + 4\pi}{0.65} \approx 27.78$ min

So during the first 20 minutes in the holding pattern, the plane is exactly 100 miles from the airport when $x \approx 6.06$ minutes, $x \approx 8.44$ minutes, $x \approx 15.72$ minutes, and $x \approx 18.11$ minutes.

c. Solve $d(x) = 70\sin(0.65x) + 150 > 100$ on the interval $[0, 20]$.

$$70\sin(0.65x) + 150 > 100$$
$$70\sin(0.65x) > -50$$
$$\sin(0.65x) > -\frac{5}{7}$$

Graphing $y_1 = \sin(0.65x)$ and $y_2 = -\frac{5}{7}$ on the interval $[0, 20]$, we see that $y_1 > y_2$ for $0 < x < 6.06$, $8.44 < x < 15.72$, and $18.11 < x < 20$.



So during the first 20 minutes in the holding pattern, the plane is more than 100 miles from the airport before 6.06 minutes, between 8.44 and 15.72 minutes, and after 18.11 minutes.

d. No, the plane is never within 70 miles of the airport while in the holding pattern. The minimum value of $\sin(0.65x)$ is -1. Thus, the least distance that the plane is from the airport is $70(-1) + 150 = 80$ miles.

108. $R(\theta) = 672\sin(2\theta)$

a. Solve $R(\theta) = 672\sin(2\theta) = 450$ on the interval $\left[0, \frac{\pi}{2}\right)$.

$$672\sin(2\theta) = 450$$
$$\sin(2\theta) = \frac{450}{672} = \frac{225}{336}$$
$$2\theta = \sin^{-1}\left(\frac{225}{336}\right) + 2k\pi$$
$$\theta = \frac{\sin^{-1}\left(\frac{225}{336}\right) + 2k\pi}{2}$$
$$\theta \approx \frac{0.7337 + 2k\pi}{2} \text{ or } \theta \approx \frac{2.408 + 2k\pi}{2},$$

k is any integer

For $k = 0$, $\theta = \frac{0.7337 + 0}{2} \approx 0.36685 \approx 21.02°$ or $\theta = \frac{2.408 + 0}{2} \approx 1.204 \approx 68.98°$

For $k = 1$, $\theta = \frac{0.7337 + 2\pi}{2} \approx 3.508 \approx 200.99°$ or $\theta = \frac{2.408 + 2\pi}{2} \approx 4.3456 \approx 248.98°$

So the golfer should hit the ball at an angle of either $21.02°$ or $68.98°$.

b. Solve $R(\theta) = 672\sin(2\theta) = 540$ on the interval $\left[0, \frac{\pi}{2}\right)$.

$$672\sin(2\theta) = 540$$

$$\sin(2\theta) = \frac{540}{672} = \frac{135}{168}$$

$$2\theta = \sin^{-1}\left(\frac{135}{168}\right) + 2k\pi$$

$$\theta = \frac{\sin^{-1}\left(\frac{135}{168}\right) + 2k\pi}{2}$$

$$\theta \approx \frac{0.9333 + 2k\pi}{2} \text{ or } \theta \approx \frac{2.2083 + 2k\pi}{2},$$

k is any integer

For $k = 0$, $\theta = \frac{0.9330 + 0}{2}$ or $\theta = \frac{2.2083 + 0}{2}$

≈ 0.46665 $\quad\approx 1.10415$

$\approx 26.74°$ $\quad\approx 63.26°$

For $k = 1$, $\theta = \frac{0.9330 + 2\pi}{2}$ or $\theta = \frac{2.2083 + 2\pi}{2}$

≈ 3.608 $\quad\approx 4.246$

$\approx 206.72°$ $\quad\approx 243.28°$

So the golfer should hit the ball at an angle of either $26.74°$ or $63.26°$.

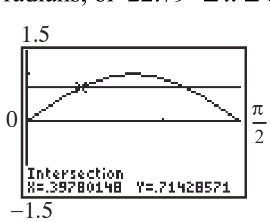
c. Solve $R(\theta) = 672\sin(2\theta) \geq 480$ on the interval $\left[0, \frac{\pi}{2}\right)$.

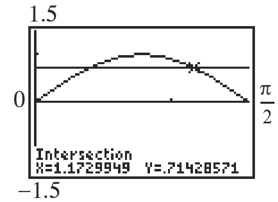
$$672\sin(2\theta) \geq 480$$

$$\sin(2\theta) \geq \frac{480}{672}$$

$$\sin(2\theta) \geq \frac{5}{7}$$

Graphing $y_1 = \sin(2x)$ and $y_2 = \frac{5}{7}$ on the interval $\left[0, \frac{\pi}{2}\right)$ and using INTERSECT, we see that $y_1 \geq y_2$ when $0.3978 \leq x \leq 1.1730$ radians, or $22.79° \leq x \leq 67.21°$.

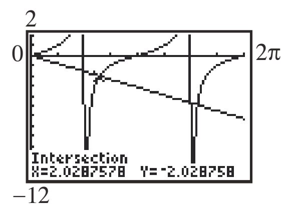


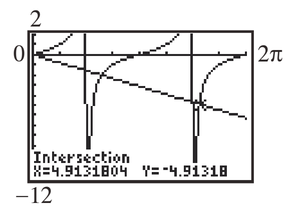


So, the golf ball will travel at least 480 feet if the angle is between about $22.79°$ and $67.21°$.

d. No; since the maximum value of the sine function is 1, the farthest the golfer can hit the ball is $672(1) = 672$ feet.

109. Find the first two positive intersection points of $Y_1 = -x$ and $Y_2 = \tan x$.





The first two positive solutions are $x \approx 2.03$ and $x \approx 4.91$.

110. a. Let L be the length of the ladder with *x* and *y* being the lengths of the two parts in each hallway.

$$L = x + y$$

$$\cos\theta = \frac{3}{x} \qquad \sin\theta = \frac{4}{y}$$

$$x = \frac{3}{\cos\theta} \qquad y = \frac{4}{\sin\theta}$$

$$L(\theta) = \frac{3}{\cos\theta} + \frac{4}{\sin\theta} = 3\sec\theta + 4\csc\theta$$

$$3\sec\theta\tan\theta - 4\csc\theta\cot\theta = 0$$

$$3\sec\theta\tan\theta = 4\csc\theta\cot\theta$$

$$\frac{\sec\theta\tan\theta}{\csc\theta\cot\theta} = \frac{4}{3}$$

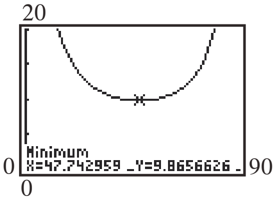
$$\tan^3\theta = \frac{4}{3}$$

$$\tan\theta = \sqrt[3]{\frac{4}{3}} \approx 1.10064$$

$$\theta \approx 47.74^{\circ}$$

b. $$L(47.74^{\circ}) = \frac{3}{\cos(47.74^{\circ})} + \frac{4}{\sin(47.74^{\circ})} \approx 9.87 \text{ feet}$$

c. Graph $Y_1 = \dfrac{3}{\cos x} + \dfrac{4}{\sin x}$ and use the MINIMUM feature:



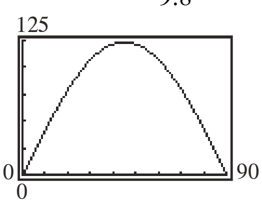
An angle of $\theta \approx 47.74°$ minimizes the length at $L \approx 9.87$ feet .

d. For this problem, only one minimum length exists. This minimum length is 9.87 feet, and it occurs when $\theta \approx 47.74°$. No matter if we find the minimum algebraically (using calculus) or graphically, the minimum will be the same.

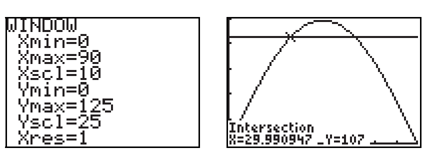
111. a.
$$107 = \frac{(34.8)^2 \sin(2\theta)}{9.8}$$
$$\sin(2\theta) = \frac{107(9.8)}{(34.8)^2} \approx 0.8659$$
$$2\theta \approx \sin^{-1}(0.8659)$$
$$2\theta \approx 60° \text{ or } 120°$$
$$\theta \approx 30° \text{ or } 60°$$

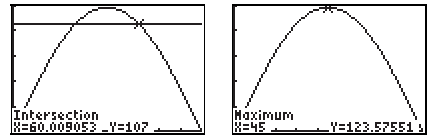
b. Notice that the answers to part (a) add up to $90°$. The maximum distance will occur when the angle of elevation is $90° \div 2 = 45°$:
$$R(45°) = \frac{(34.8)^2 \sin\left[2(45°)\right]}{9.8} \approx 123.6$$
The maximum distance is 123.6 meters.

c. Let $Y_1 = \dfrac{(34.8)^2 \sin(2x)}{9.8}$



d.

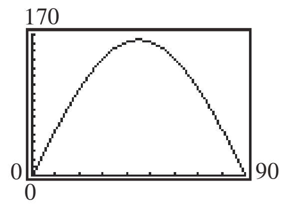




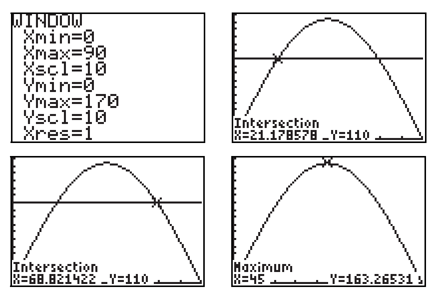
112. a.
$$110 = \frac{(40)^2 \sin(2\theta)}{9.8}$$
$$\sin(2\theta) = \frac{110 \cdot 9.8}{40^2} \approx 0.67375$$
$$2\theta \approx \sin^{-1}(0.67375)$$
$$2\theta \approx 42.4° \text{ or } 137.6°$$
$$\theta \approx 21.2° \text{ or } 68.8°$$

b. The maximum distance will occur when the angle of elevation is $45°$:
$$R(45°) = \frac{(40)^2 \sin\left[2(45°)\right]}{9.8} \approx 163.3$$
The maximum distance is approximately 163.3 meter

c. Let $Y_1 = \dfrac{(40)^2 \sin(2x)}{9.8}$:



d.



113.
$$\frac{\sin 40°}{\sin \theta_2} = 1.33$$
$$1.33 \sin \theta_2 = \sin 40°$$
$$\sin \theta_2 = \frac{\sin 40°}{1.33} \approx 0.4833$$
$$\theta_2 = \sin^{-1}(0.4833) \approx 28.90°$$

114. $\frac{\sin 50°}{\sin\theta_2} = 1.66$

$1.66\sin\theta_2 = \sin 50°$

$\sin\theta_2 = \frac{\sin 50°}{1.66} \approx 0.4615$

$\theta_2 = \sin^{-1}(0.4615) \approx 27.48°$

115. Calculate the index of refraction for each:

θ_1	θ_2	$\frac{v_1}{v_2} = \frac{\sin\theta_1}{\sin\theta_2}$
10º	8º	$\frac{\sin 10^o}{\sin 8^o} \approx 1.2477$
20º	$15^o 30' = 15.5^o$	$\frac{\sin 20^o}{\sin 15.5^o} \approx 1.2798$
30º	$22^o 30' = 22.5^o$	$\frac{\sin 30^o}{\sin 22.5^o} \approx 1.3066$
40º	$29^o 0' = 29^o$	$\frac{\sin 40^o}{\sin 29^o} \approx 1.3259$
50º	$35^o 0' = 35^o$	$\frac{\sin 50^o}{\sin 35^o} \approx 1.3356$
60º	$40^o 30' = 40.5^o$	$\frac{\sin 60^o}{\sin 40.5^o} \approx 1.3335$
70º	$45^o 30' = 45.5^o$	$\frac{\sin 70^o}{\sin 45.5^o} \approx 1.3175$
80º	$50^o 0' = 50^o$	$\frac{\sin 80^o}{\sin 50^o} \approx 1.2856$

Yes, these data values agree with Snell's Law. The results vary from about 1.25 to 1.34.

116. $\frac{v_1}{v_2} = \frac{2.998\times10^8}{1.92\times10^8} \approx 1.56$

The index of refraction for this liquid is about 1.56.

117. Calculate the index of refraction:

$\theta_1 = 40^o, \theta_2 = 26^o; \quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin 40^o}{\sin 26^o} \approx 1.47$

118. The index of refraction of crown glass is 1.52.

$\frac{\sin 30^o}{\sin\theta_2} \approx 1.52$

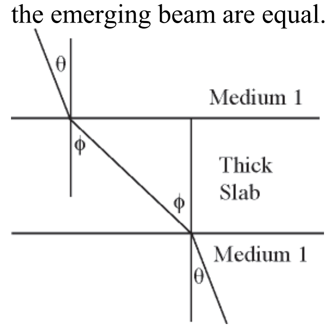
$1.52\sin\theta_2 = \sin 30°$

$\sin\theta_2 = \frac{\sin 30°}{1.52} \approx 0.3289$

$\theta_2 \approx \sin^{-1}(0.3352) \approx 19.20°$

The angle of refraction is about $19.20°$.

119. If θ is the original angle of incidence and ϕ is the angle of refraction, then $\frac{\sin\theta}{\sin\phi} = n_2$. The angle of incidence of the emerging beam is also ϕ, and the index of refraction is $\frac{1}{n_2}$. Thus, θ is the angle of refraction of the emerging beam. The two beams are parallel since the original angle of incidence and the angle of refraction of the emerging beam are equal.



120. Here we have $n_1 = 1.33$ and $n_2 = 1.52$.

$n_1 \sin\theta_B = n_2 \cos_B$

$\frac{\sin\theta_B}{\cos\theta_B} = \frac{n_2}{n_1}$

$\tan\theta_B = \frac{n_2}{n_1}$

$\theta_B = \tan^{-1}\frac{n_2}{n_1} = \tan^{-1}\left(\frac{1.52}{1.33}\right) \approx 48.8°$

121. Answers will vary.

122. Since the range of $y = \sin x$ is $-1 \le y \le 1$, then $y = 5\sin x + x$ cannot be equal to 3 when $x > 4\pi$ or $x < -\pi$ since you are multiplying the result by 5 and adding x.

123. $6^x = y \leftrightarrow x = \log_6 y$

124. $x = \frac{-(-9)\pm\sqrt{(-9)^2 - 4(2)(8)}}{2(2)}$

$= \frac{9\pm\sqrt{81-64}}{4}$

$= \frac{9\pm\sqrt{17}}{4}$

So the solution set is: $\left\{\frac{9-\sqrt{17}}{4}, \frac{9+\sqrt{17}}{4}\right\}$.

125. $\sin\theta = -\dfrac{\sqrt{10}}{10}, \quad \cos\theta = \dfrac{3\sqrt{10}}{10}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(-\frac{\sqrt{10}}{10}\right)}{\left(\frac{3\sqrt{10}}{10}\right)} = -\frac{\sqrt{10}}{10}\cdot\frac{10}{3\sqrt{10}} = -\frac{1}{3}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\left(-\frac{\sqrt{10}}{10}\right)} = 1\left(-\frac{10}{\sqrt{10}}\right)\frac{\sqrt{10}}{\sqrt{10}} = -\sqrt{10}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left(\frac{3\sqrt{10}}{10}\right)} = \frac{10}{3\sqrt{10}}\cdot\frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{3}$$

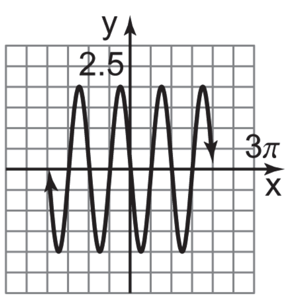
$$\cot\theta = \frac{1}{\tan\theta} = -3$$

126. $y = 2\sin(2x - \pi)$

Amplitude: $|A| = |2| = 2$

Period: $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$

Phase Shift: $\dfrac{\phi}{\omega} = \dfrac{\pi}{2} = \dfrac{\pi}{2}$



Section 7.4

1. True
2. True
3. identity; conditional
4. −1
5. 0
6. True
7. False, you need to work with one side only.
8. True
9. c
10. b

11. $\tan\theta\cdot\csc\theta = \dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\sin\theta} = \dfrac{1}{\cos\theta}$

12. $\cot\theta\cdot\sec\theta = \dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{1}{\cos\theta} = \dfrac{1}{\sin\theta}$

13.
$$\frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta} = \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta} = \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta} = \frac{1+\sin\theta}{\cos\theta}$$

14.
$$\frac{\sin\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta} = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} = \frac{1-\cos\theta}{\sin\theta}$$

15.
$$\frac{\sin\theta+\cos\theta}{\cos\theta} + \frac{\cos\theta-\sin\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta + \sin\theta\cos\theta + \cos\theta(\cos\theta - \sin\theta)}{\sin\theta\cos\theta}$$
$$= \frac{\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta - \cos\theta\sin\theta}{\sin\theta\cos\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta + \sin\theta\cos\theta - \cos\theta\sin\theta}{\sin\theta\cos\theta}$$
$$= \frac{1}{\sin\theta\cos\theta}$$

16.
$$\frac{1}{1-\cos v} + \frac{1}{1+\cos v} = \frac{1+\cos v + 1 - \cos v}{(1-\cos v)(1+\cos v)} = \frac{2}{1-\cos^2 v} = \frac{2}{\sin^2 v}$$

17. $\dfrac{(\sin\theta+\cos\theta)(\sin\theta+\cos\theta)-1}{\sin\theta\cos\theta}$
$=\dfrac{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta-1}{\sin\theta\cos\theta}$
$=\dfrac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$
$=\dfrac{1+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$
$=\dfrac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$
$=2$

18. $\dfrac{(\tan\theta+1)(\tan\theta+1)-\sec^2\theta}{\tan\theta}$
$=\dfrac{\tan^2\theta+2\tan\theta+1-\sec^2\theta}{\tan\theta}$
$=\dfrac{\tan^2\theta+1+2\tan\theta-\sec^2\theta}{\tan\theta}$
$=\dfrac{\sec^2\theta+2\tan\theta-\sec^2\theta}{\tan\theta}$
$=\dfrac{2\tan\theta}{\tan\theta}$
$=2$

19. $\dfrac{3\sin^2\theta+4\sin\theta+1}{\sin^2\theta+2\sin\theta+1}=\dfrac{(3\sin\theta+1)(\sin\theta+1)}{(\sin\theta+1)(\sin\theta+1)}$
$=\dfrac{3\sin\theta+1}{\sin\theta+1}$

20. $\dfrac{\cos^2\theta-1}{\cos^2\theta-\cos\theta}=\dfrac{(\cos\theta+1)(\cos\theta-1)}{\cos\theta(\cos\theta-1)}$
$=\dfrac{\cos\theta+1}{\cos\theta}$

21. $\csc\theta\cdot\cos\theta=\dfrac{1}{\sin\theta}\cdot\cos\theta=\dfrac{\cos\theta}{\sin\theta}=\cot\theta$

22. $\sec\theta\cdot\sin\theta=\dfrac{1}{\cos\theta}\cdot\sin\theta=\dfrac{\sin\theta}{\cos\theta}=\tan\theta$

23. $1+\tan^2(-\theta)=1+(-\tan\theta)^2=1+\tan^2\theta=\sec^2\theta$

24. $1+\cot^2(-\theta)=1+(-\cot\theta)^2=1+\cot^2\theta=\csc^2\theta$

25. $\cos\theta(\tan\theta+\cot\theta)=\cos\theta\left(\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}\right)$
$=\cos\theta\left(\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)$
$=\cos\theta\left(\dfrac{1}{\cos\theta\sin\theta}\right)$
$=\dfrac{1}{\sin\theta}$
$=\csc\theta$

26. $\sin\theta(\cot\theta+\tan\theta)=\sin\theta\left(\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\cos\theta}\right)$
$=\sin\theta\left(\dfrac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}\right)$
$=\sin\theta\left(\dfrac{1}{\sin\theta\cos\theta}\right)$
$=\dfrac{1}{\cos\theta}$
$=\sec\theta$

27. $\tan u\cot u-\cos^2 u=\tan u\cdot\dfrac{1}{\tan u}-\cos^2 u$
$=1-\cos^2 u$
$=\sin^2 u$

28. $\sin u\csc u-\cos^2 u=\sin u\cdot\dfrac{1}{\sin u}-\cos^2 u$
$=1-\cos^2 u$
$=\sin^2 u$

29. $(\sec\theta-1)(\sec\theta+1)=\sec^2\theta-1=\tan^2\theta$

30. $(\csc\theta-1)(\csc\theta+1)=\csc^2\theta-1=\cot^2\theta$

31. $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=\sec^2\theta-\tan^2\theta=1$

32. $(\csc\theta+\cot\theta)(\csc\theta-\cot\theta)=\csc^2\theta-\cot^2\theta=1$

33. $\cos^2\theta(1+\tan^2\theta)=\cos^2\theta\cdot\sec^2\theta$
$=\cos^2\theta\cdot\dfrac{1}{\cos^2\theta}$
$=1$

34. $(1-\cos^2\theta)(1+\cot^2\theta)=\sin^2\theta\cdot\csc^2\theta$
$$=\sin^2\theta\cdot\frac{1}{\sin^2\theta}$$
$$=1$$

35. $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2$
$$=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta$$
$$+\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta$$
$$=2\sin^2\theta+2\cos^2\theta$$
$$=2(\sin^2\theta+\cos^2\theta)$$
$$=2\cdot 1$$
$$=2$$

36. $\tan^2\theta\cos^2\theta+\cot^2\theta\sin^2\theta$
$$=\frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta+\frac{\cos^2\theta}{\sin^2\theta}\cdot\sin^2\theta$$
$$=\sin^2\theta+\cos^2\theta$$
$$=1$$

37. $\sec^4\theta-\sec^2\theta=\sec^2\theta(\sec^2\theta-1)$
$$=(\tan^2\theta+1)\tan^2\theta$$
$$=\tan^4\theta+\tan^2\theta$$

38. $\csc^4\theta-\csc^2\theta=\csc^2\theta(\csc^2\theta-1)$
$$=(\cot^2\theta+1)\cot^2\theta$$
$$=\cot^4\theta+\cot^2\theta$$

39. $\sec u-\tan u=\dfrac{1}{\cos u}-\dfrac{\sin u}{\cos u}$
$$=\left(\frac{1-\sin u}{\cos u}\right)\cdot\left(\frac{1+\sin u}{1+\sin u}\right)$$
$$=\frac{1-\sin^2 u}{\cos u(1+\sin u)}$$
$$=\frac{\cos^2 u}{\cos u(1+\sin u)}$$
$$=\frac{\cos u}{1+\sin u}$$

40. $\csc u-\cot u=\dfrac{1}{\sin u}-\dfrac{\cos u}{\sin u}$
$$=\left(\frac{1-\cos u}{\sin u}\right)\cdot\left(\frac{1+\cos u}{1+\cos u}\right)$$
$$=\frac{1-\cos^2 u}{\sin u(1+\cos u)}$$
$$=\frac{\sin^2 u}{\sin u(1+\cos u)}$$
$$=\frac{\sin u}{1+\cos u}$$

41. $3\sin^2\theta+4\cos^2\theta=3\sin^2\theta+3\cos^2\theta+\cos^2\theta$
$$=3(\sin^2\theta+\cos^2\theta)+\cos^2\theta$$
$$=3\cdot 1+\cos^2\theta$$
$$=3+\cos^2\theta$$

42. $9\sec^2\theta-5\tan^2\theta=4\sec^2\theta+5\sec^2\theta-5\tan^2\theta$
$$=4\sec^2\theta+5(\sec^2\theta-\tan^2\theta)$$
$$=4\sec^2\theta+5\cdot 1$$
$$=5+4\sec^2\theta$$

43. $1-\dfrac{\cos^2\theta}{1+\sin\theta}=1-\dfrac{1-\sin^2\theta}{1+\sin\theta}$
$$=1-\frac{(1-\sin\theta)(1+\sin\theta)}{1+\sin\theta}$$
$$=1-(1-\sin\theta)$$
$$=1-1+\sin\theta$$
$$=\sin\theta$$

44. $1-\dfrac{\sin^2\theta}{1-\cos\theta}=1-\dfrac{1-\cos^2\theta}{1-\cos\theta}$
$$=1-\frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta}$$
$$=1-(1+\cos\theta)$$
$$=1-1-\cos\theta$$
$$=-\cos\theta$$

45. $\dfrac{1+\tan v}{1-\tan v}=\dfrac{1+\dfrac{1}{\cot v}}{1-\dfrac{1}{\cot v}}$
$$=\frac{\left(1+\dfrac{1}{\cot v}\right)\cot v}{\left(1-\dfrac{1}{\cot v}\right)\cot v}$$
$$=\frac{\cot v+1}{\cot v-1}$$

46. $$\frac{\csc v-1}{\csc v+1}=\frac{\frac{1}{\sin v}-1}{\frac{1}{\sin v}+1}=\frac{\left(\frac{1}{\sin v}-1\right)\sin v}{\left(\frac{1}{\sin v}+1\right)\sin v}=\frac{1-\sin v}{1+\sin v}$$

47. $$\frac{\sec\theta}{\csc\theta}+\frac{\sin\theta}{\cos\theta}=\frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}}+\frac{\sin\theta}{\cos\theta}=\frac{\sin\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\tan\theta+\tan\theta=2\tan\theta$$

48. $$\frac{\csc\theta-1}{\cot\theta}=\frac{\csc\theta-1}{\cot\theta}\cdot\frac{\csc\theta+1}{\csc\theta+1}=\frac{\csc^2\theta-1}{\cot\theta(\csc\theta+1)}=\frac{\cot^2\theta}{\cot\theta(\csc\theta+1)}=\frac{\cot\theta}{\csc\theta+1}$$

49. $$\frac{1+\sin\theta}{1-\sin\theta}=\frac{1+\frac{1}{\csc\theta}}{1-\frac{1}{\csc\theta}}=\frac{\frac{\csc\theta+1}{\csc\theta}}{\frac{\csc\theta-1}{\csc\theta}}=\frac{\csc\theta+1}{\csc\theta}\cdot\frac{\csc\theta}{\csc\theta-1}=\frac{\csc\theta+1}{\csc\theta-1}$$

50. $$\frac{\cos\theta+1}{\cos\theta-1}=\frac{\frac{1}{\sec\theta}+1}{\frac{1}{\sec\theta}-1}=\frac{\frac{1+\sec\theta}{\sec\theta}}{\frac{1-\sec\theta}{\sec\theta}}=\frac{1+\sec\theta}{1-\sec\theta}$$

51. $$\frac{1-\sin v}{\cos v}+\frac{\cos v}{1-\sin v}=\frac{(1-\sin v)^2+\cos^2 v}{\cos v(1-\sin v)}=\frac{1-2\sin v+\sin^2 v+\cos^2 v}{\cos v(1-\sin v)}=\frac{1-2\sin v+1}{\cos v(1-\sin v)}=\frac{2-2\sin v}{\cos v(1-\sin v)}=\frac{2(1-\sin v)}{\cos v(1-\sin v)}=\frac{2}{\cos v}=2\sec v$$

52. $$\frac{\cos v}{1+\sin v}+\frac{1+\sin v}{\cos v}=\frac{\cos^2 v+(1+\sin v)^2}{\cos v(1+\sin v)}=\frac{\cos^2 v+1+2\sin v+\sin^2 v}{\cos v(1+\sin v)}=\frac{2+2\sin v}{\cos v(1+\sin v)}=\frac{2(1+\sin v)}{\cos v(1+\sin v)}=\frac{2}{\cos v}=2\sec v$$

53. $$\frac{\sin\theta}{\sin\theta-\cos\theta}=\frac{\sin\theta}{\sin\theta-\cos\theta}\cdot\frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta}}=\frac{1}{1-\frac{\cos\theta}{\sin\theta}}=\frac{1}{1-\cot\theta}$$

54. $1-\dfrac{\sin^2\theta}{1+\cos\theta}=1-\dfrac{1-\cos^2\theta}{1+\cos\theta}$

$$=1-\frac{(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta}$$
$$=1-(1-\cos\theta)$$
$$=\cos\theta$$

55. $(\sec\theta-\tan\theta)^2$

$$=\sec^2\theta-2\sec\theta\tan\theta+\tan^2\theta$$
$$=\frac{1}{\cos^2\theta}-2\cdot\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}+\frac{\sin^2\theta}{\cos^2\theta}$$
$$=\frac{1-2\sin\theta+\sin^2\theta}{\cos^2\theta}$$
$$=\frac{(1-\sin\theta)(1-\sin\theta)}{1-\sin^2\theta}$$
$$=\frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$
$$=\frac{1-\sin\theta}{1+\sin\theta}$$

56. $(\csc\theta-\cot\theta)^2$

$$=\csc^2\theta-2\csc\theta\cot\theta+\cot^2\theta$$
$$=\frac{1}{\sin^2\theta}-2\cdot\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}+\frac{\cos^2\theta}{\sin^2\theta}$$
$$=\frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}$$
$$=\frac{(1-\cos\theta)(1-\cos\theta)}{1-\cos^2\theta}$$
$$=\frac{(1-\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$
$$=\frac{1-\cos\theta}{1+\cos\theta}$$

57. $\dfrac{\cos\theta}{1-\tan\theta}+\dfrac{\sin\theta}{1-\cot\theta}$

$$=\frac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}}+\frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}$$
$$=\frac{\cos\theta}{\frac{\cos\theta-\sin\theta}{\cos\theta}}+\frac{\sin\theta}{\frac{\sin\theta-\cos\theta}{\sin\theta}}$$
$$=\frac{\cos^2\theta}{\cos\theta-\sin\theta}+\frac{\sin^2\theta}{\sin\theta-\cos\theta}$$
$$=\frac{\cos^2\theta-\sin^2\theta}{\cos\theta-\sin\theta}$$
$$=\frac{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{\cos\theta-\sin\theta}$$
$$=\sin\theta+\cos\theta$$

58. $\dfrac{\cot\theta}{1-\tan\theta}+\dfrac{\tan\theta}{1-\cot\theta}$

$$=\frac{\frac{\cos\theta}{\sin\theta}}{1-\frac{\sin\theta}{\cos\theta}}+\frac{\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}}$$
$$=\frac{\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}+\frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta-\cos\theta}{\sin\theta}}$$
$$=\frac{\cos^2\theta}{\sin\theta(\cos\theta-\sin\theta)}+\frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)}$$
$$=\frac{-\cos^2\theta\cdot\cos\theta+\sin^2\theta\cdot\sin\theta}{\sin\theta\cos\theta(\sin\theta-\cos\theta)}$$
$$=\frac{\sin^3\theta-\cos^3\theta}{\sin\theta\cos\theta(\sin\theta-\cos\theta)}$$
$$=\frac{(\sin\theta-\cos\theta)(\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta\cos\theta(\sin\theta-\cos\theta)}$$
$$=\frac{\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin^2\theta}{\sin\theta\cos\theta}+\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}+\frac{\cos^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin\theta}{\cos\theta}+1+\frac{\cos\theta}{\sin\theta}$$
$$=1+\tan\theta+\cot\theta$$

59. $\tan\theta + \dfrac{\cos\theta}{1+\sin\theta} = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta}$

$$= \frac{\sin\theta(1+\sin\theta)+\cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{\sin\theta+1}{\cos\theta(1+\sin\theta)}$$
$$= \frac{1}{\cos\theta}$$
$$= \sec\theta$$

60. $\dfrac{\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta} = \dfrac{(\sin\theta\cos\theta)\cdot\dfrac{1}{\cos^2\theta}}{(\cos^2\theta-\sin^2\theta)\cdot\dfrac{1}{\cos^2\theta}}$

$$= \frac{\dfrac{\sin\theta}{\cos\theta}}{1-\dfrac{\sin^2\theta}{\cos^2\theta}}$$
$$= \frac{\tan\theta}{1-\tan^2\theta}$$

61. $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$

$$= \frac{\tan\theta+(\sec\theta-1)}{\tan\theta-(\sec\theta-1)}\cdot\frac{\tan\theta+(\sec\theta-1)}{\tan\theta+(\sec\theta-1)}$$
$$= \frac{\tan^2\theta+2\tan\theta(\sec\theta-1)+\sec^2\theta-2\sec\theta+1}{\tan^2\theta-(\sec^2\theta-2\sec\theta+1)}$$
$$= \frac{\sec^2\theta-1+2\tan\theta(\sec\theta-1)+\sec^2\theta-2\sec\theta+1}{\sec^2\theta-1-\sec^2\theta+2\sec\theta-1}$$
$$= \frac{2\sec^2\theta-2\sec\theta+2\tan\theta(\sec\theta-1)}{2\sec\theta-2}$$
$$= \frac{2\sec\theta(\sec\theta-1)+2\tan\theta(\sec\theta-1)}{2\sec\theta-2}$$
$$= \frac{2(\sec\theta-1)(\sec\theta+\tan\theta)}{2(\sec\theta-1)}$$
$$= \tan\theta+\sec\theta$$

62. $\dfrac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1}$

$$= \frac{(\sin\theta-\cos\theta)+1}{(\sin\theta+\cos\theta)-1}\cdot\frac{(\sin\theta+\cos\theta)+1}{(\sin\theta+\cos\theta)+1}$$
$$= \frac{\sin^2\theta-\cos^2\theta+\sin\theta+\cos\theta+\sin\theta-\cos\theta+1}{(\sin\theta+\cos\theta)^2-1}$$
$$= \frac{\sin^2\theta-\cos^2\theta+2\sin\theta+1}{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta-1}$$
$$= \frac{\sin^2\theta-(1-\sin^2\theta)+2\sin\theta+1}{2\sin\theta\cos\theta+1-1}$$
$$= \frac{2\sin^2\theta+2\sin\theta}{2\sin\theta\cos\theta}$$
$$= \frac{2\sin\theta(\sin\theta+1)}{2\sin\theta\cos\theta}$$
$$= \frac{\sin\theta+1}{\cos\theta}$$

63. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta} = \dfrac{\dfrac{\sin\theta}{\cos\theta}-\dfrac{\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}}$

$$= \frac{\dfrac{\sin^2\theta-\cos^2\theta}{\cos\theta\sin\theta}}{\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}}$$
$$= \frac{\sin^2\theta-\cos^2\theta}{1}$$
$$= \sin^2\theta-\cos^2\theta$$

64. $\dfrac{\sec\theta-\cos\theta}{\sec\theta+\cos\theta} = \dfrac{\dfrac{1}{\cos\theta}-\dfrac{\cos^2\theta}{\cos\theta}}{\dfrac{1}{\cos\theta}+\dfrac{\cos^2\theta}{\cos\theta}}$

$$= \frac{\dfrac{1-\cos^2\theta}{\cos\theta}}{\dfrac{1+\cos^2\theta}{\cos\theta}}$$
$$= \frac{1-\cos^2\theta}{1+\cos^2\theta}$$
$$= \frac{\sin^2\theta}{1+\cos^2\theta}$$

65. $$\frac{\tan u-\cot u}{\tan u+\cot u}+1=\frac{\frac{\sin u}{\cos u}-\frac{\cos u}{\sin u}}{\frac{\sin u}{\cos u}+\frac{\cos u}{\sin u}}+1$$
$$=\frac{\frac{\sin^2 u-\cos^2 u}{\cos u\sin u}}{\frac{\sin^2 u+\cos^2 u}{\cos u\sin u}}+1$$
$$=\frac{\sin^2 u-\cos^2 u}{1}+1$$
$$=\sin^2 u-\cos^2 u+1$$
$$=\sin^2 u+(1-\cos^2 u)$$
$$=\sin^2 u+\sin^2 u$$
$$=2\sin^2 u$$

66. $$\frac{\tan u-\cot u}{\tan u+\cot u}+2\cos^2 u=\frac{\frac{\sin u}{\cos u}-\frac{\cos u}{\sin u}}{\frac{\sin u}{\cos u}+\frac{\cos u}{\sin u}}+2\cos^2 u$$
$$=\frac{\frac{\sin^2 u-\cos^2 u}{\cos u\sin u}}{\frac{\sin^2 u+\cos^2 u}{\cos u\sin u}}+2\cos^2 u$$
$$=\frac{\sin^2 u-\cos^2 u}{1}+2\cos^2 u$$
$$=\sin^2 u+\cos^2 u$$
$$=1$$

67. $$\frac{\sec\theta+\tan\theta}{\cot\theta+\cos\theta}=\frac{\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta}+\cos\theta}$$
$$=\frac{\frac{1+\sin\theta}{\cos\theta}}{\frac{\cos\theta+\cos\theta\sin\theta}{\sin\theta}}$$
$$=\frac{1+\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta(1+\sin\theta)}$$
$$=\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$$
$$=\tan\theta\sec\theta$$

68. $$\frac{\sec\theta}{1+\sec\theta}=\frac{\frac{1}{\cos\theta}}{1+\frac{1}{\cos\theta}}$$
$$=\frac{\frac{1}{\cos\theta}}{\frac{\cos\theta+1}{\cos\theta}}$$
$$=\left(\frac{1}{1+\cos\theta}\right)\cdot\left(\frac{1-\cos\theta}{1-\cos\theta}\right)$$
$$=\frac{1-\cos\theta}{1-\cos^2\theta}$$
$$=\frac{1-\cos\theta}{\sin^2\theta}$$

69. $$\frac{1-\tan^2\theta}{1+\tan^2\theta}+1=\frac{1-\tan^2\theta}{1+\tan^2\theta}+\frac{1+\tan^2\theta}{1+\tan^2\theta}$$
$$=\frac{1-\tan^2\theta+1+\tan^2\theta}{1+\tan^2\theta}$$
$$=\frac{2}{1+\tan^2\theta}=\frac{2}{\sec^2\theta}$$
$$=2\cdot\frac{1}{\sec^2\theta}$$
$$=2\cos^2\theta$$

70. $$\frac{1-\cot^2\theta}{1+\cot^2\theta}+2\cos^2\theta=\frac{1-\cot^2\theta}{\csc^2\theta}+2\cos^2\theta$$
$$=\frac{1}{\csc^2\theta}-\frac{\cot^2\theta}{\csc^2\theta}+2\cos^2\theta$$
$$=\sin^2\theta-\frac{\frac{\cos^2\theta}{\sin^2\theta}}{\frac{1}{\sin^2\theta}}+2\cos^2\theta$$
$$=\sin^2\theta-\cos^2\theta+2\cos^2\theta$$
$$=\sin^2\theta+\cos^2\theta$$
$$=1$$

71. $$\frac{\sec\theta-\csc\theta}{\sec\theta\csc\theta}=\frac{\sec\theta}{\sec\theta\csc\theta}-\frac{\csc\theta}{\sec\theta\csc\theta}$$
$$=\frac{1}{\csc\theta}-\frac{1}{\sec\theta}$$
$$=\sin\theta-\cos\theta$$

72. $\dfrac{\sin^2\theta-\tan\theta}{\cos^2\theta-\cot\theta}$

$$=\frac{\sin^2\theta-\dfrac{\sin\theta}{\cos\theta}}{\cos^2\theta-\dfrac{\cos\theta}{\sin\theta}}$$

$$=\frac{\dfrac{\sin^2\theta\cos\theta-\sin\theta}{\cos\theta}}{\dfrac{\cos^2\theta\sin\theta-\cos\theta}{\sin\theta}}$$

$$=\frac{\sin^2\theta\cos\theta-\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\cos^2\theta\sin\theta-\cos\theta}$$

$$=\frac{\sin\theta(\sin\theta\cos\theta-1)}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta(\cos\theta\sin\theta-1)}$$

$$=\frac{\sin^2\theta}{\cos^2\theta}$$

$$=\tan^2\theta$$

73. $\sec\theta-\cos\theta=\dfrac{1}{\cos\theta}-\cos\theta$

$$=\frac{1-\cos^2\theta}{\cos\theta}$$

$$=\frac{\sin^2\theta}{\cos\theta}$$

$$=\sin\theta\cdot\frac{\sin\theta}{\cos\theta}$$

$$=\sin\theta\tan\theta$$

74. $\tan\theta+\cot\theta=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}$

$$=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$

$$=\frac{1}{\sin\theta\cos\theta}$$

$$=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}$$

$$=\sec\theta\csc\theta$$

75. $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=\dfrac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)}$

$$=\frac{2}{1-\sin^2\theta}$$

$$=\frac{2}{\cos^2\theta}$$

$$=2\sec^2\theta$$

76. $\dfrac{1+\sin\theta}{1-\sin\theta}-\dfrac{1-\sin\theta}{1+\sin\theta}$

$$=\frac{(1+\sin\theta)^2-(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$$

$$=\frac{1+2\sin\theta+\sin^2\theta-(1-2\sin\theta+\sin^2\theta)}{1-\sin^2\theta}$$

$$=\frac{4\sin\theta}{\cos^2\theta}$$

$$=4\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$$

$$=4\tan\theta\sec\theta$$

77. $\dfrac{\sec\theta}{1-\sin\theta}=\left(\dfrac{\sec\theta}{1-\sin\theta}\right)\cdot\left(\dfrac{1+\sin\theta}{1+\sin\theta}\right)$

$$=\frac{\sec\theta(1+\sin\theta)}{1-\sin^2\theta}$$

$$=\frac{\sec\theta(1+\sin\theta)}{\cos^2\theta}$$

$$=\frac{1}{\cos\theta}\cdot\frac{1+\sin\theta}{\cos^2\theta}$$

$$=\frac{1+\sin\theta}{\cos^3\theta}$$

78. $\dfrac{1+\sin\theta}{1-\sin\theta}=\dfrac{(1+\sin\theta)(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$

$$=\frac{(1+\sin\theta)^2}{1-\sin^2\theta}$$

$$=\frac{(1+\sin\theta)^2}{\cos^2\theta}$$

$$=\left(\frac{1+\sin\theta}{\cos\theta}\right)^2$$

$$=\left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)^2$$

$$=(\sec\theta+\tan\theta)^2$$

79. $\dfrac{(\sec v-\tan v)^2+1}{\csc v(\sec v-\tan v)}$

$=\dfrac{\sec^2 v-2\sec v\tan v+\tan^2 v+1}{\csc v(\sec v-\tan v)}$

$=\dfrac{\sec^2 v-2\sec v\tan v+\sec^2 v}{\csc v(\sec v-\tan v)}$

$=\dfrac{2\sec^2 v-2\sec v\tan v}{\csc v(\sec v-\tan v)}$

$=\dfrac{2\sec v(\sec v-\tan v)}{\csc v(\sec v-\tan v)}$

$=\dfrac{2\sec v}{\csc v}$

$=\dfrac{2\cdot\dfrac{1}{\cos v}}{\dfrac{1}{\sin v}}$

$=2\cdot\dfrac{1}{\cos v}\cdot\dfrac{\sin v}{1}$

$=2\cdot\dfrac{\sin v}{\cos v}$

$=2\tan v$

80. $\dfrac{\sec^2 v-\tan^2 v+\tan v}{\sec v}=\dfrac{1+\tan v}{\sec v}$

$=\dfrac{1+\dfrac{\sin v}{\cos v}}{\dfrac{1}{\cos v}}$

$=\dfrac{\dfrac{\cos v+\sin v}{\cos v}}{\dfrac{1}{\cos v}}$

$=\cos v+\sin v$

81. $\dfrac{\sin\theta+\cos\theta}{\cos\theta}-\dfrac{\sin\theta-\cos\theta}{\sin\theta}$

$=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\cos\theta}-\dfrac{\sin\theta}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}$

$=\dfrac{\sin\theta}{\cos\theta}+1-1+\dfrac{\cos\theta}{\sin\theta}$

$=\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}$

$=\dfrac{1}{\cos\theta\sin\theta}$

$=\sec\theta\csc\theta$

82. $\dfrac{\sin\theta+\cos\theta}{\sin\theta}-\dfrac{\cos\theta-\sin\theta}{\cos\theta}$

$=\dfrac{\sin\theta}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}-\dfrac{\cos\theta}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}$

$=1+\dfrac{\cos\theta}{\sin\theta}-1+\dfrac{\sin\theta}{\cos\theta}$

$=\dfrac{\cos^2\theta+\sin^2\theta}{\cos\theta\sin\theta}$

$=\dfrac{1}{\cos\theta\sin\theta}$

$=\sec\theta\csc\theta$

83. $\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}$

$=\dfrac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta}$

$=\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta$

$=1-\sin\theta\cos\theta$

84. $\dfrac{\sin^3\theta+\cos^3\theta}{1-2\cos^2\theta}$

$=\dfrac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{1-\cos^2\theta-\cos^2\theta}$

$=\dfrac{(\sin\theta+\cos\theta)(\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta)}{\sin^2\theta-\cos^2\theta}$

$=\dfrac{(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)}{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}$

$=\dfrac{1-\sin\theta\cos\theta}{\sin\theta-\cos\theta}\cdot\dfrac{\dfrac{1}{\cos\theta}}{\dfrac{1}{\cos\theta}}$

$=\dfrac{\dfrac{1}{\cos\theta}-\sin\theta}{\dfrac{\sin\theta}{\cos\theta}-1}$

$=\dfrac{\sec\theta-\sin\theta}{\tan\theta-1}$

85. $\dfrac{\cos^2\theta-\sin^2\theta}{1-\tan^2\theta}=\dfrac{\cos^2\theta-\sin^2\theta}{1-\dfrac{\sin^2\theta}{\cos^2\theta}}$

$=\dfrac{\cos^2\theta-\sin^2\theta}{\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}$

$=\left(\cos^2\theta-\sin^2\theta\right)\cdot\dfrac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}$

$=\cos^2\theta$

86. $\dfrac{\cos\theta+\sin\theta-\sin^3\theta}{\sin\theta}=\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\sin\theta}-\dfrac{\sin^3\theta}{\sin\theta}$
$=\cot\theta+1-\sin^2\theta$
$=\cot\theta+\cos^2\theta$

87. $\dfrac{(2\cos^2\theta-1)^2}{\cos^4\theta-\sin^4\theta}=\dfrac{\left[2\cos^2\theta-(\sin^2\theta+\cos^2\theta)\right]^2}{(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)}$
$=\dfrac{(\cos^2\theta-\sin^2\theta)^2}{(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)}$
$=\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$
$=\cos^2\theta-\sin^2\theta$
$=1-\sin^2\theta-\sin^2\theta$
$=1-2\sin^2\theta$

88. $\dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta}=\dfrac{1-\cos^2\theta-\cos^2\theta}{\sin\theta\cos\theta}$
$=\dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}$
$=\dfrac{\sin^2\theta}{\sin\theta\cos\theta}-\dfrac{\cos^2\theta}{\sin\theta\cos\theta}$
$=\dfrac{\sin\theta}{\cos\theta}-\dfrac{\cos\theta}{\sin\theta}$
$=\tan\theta-\cot\theta$

89. $\dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$
$=\dfrac{(1+\sin\theta)+\cos\theta}{(1+\sin\theta)-\cos\theta}\cdot\dfrac{(1+\sin\theta)+\cos\theta}{(1+\sin\theta)+\cos\theta}$
$=\dfrac{1+2\sin\theta+\sin^2\theta+2\cos\theta(1+\sin\theta)+\cos^2\theta}{1+2\sin\theta+\sin^2\theta-\cos^2\theta}$
$=\dfrac{1+2\sin\theta+\sin^2\theta+2\cos\theta(1+\sin\theta)+(1-\sin^2\theta)}{1+2\sin\theta+\sin^2\theta-(1-\sin^2\theta)}$
$=\dfrac{2+2\sin\theta+2\cos\theta(1+\sin\theta)}{2\sin\theta+2\sin^2\theta}$
$=\dfrac{2(1+\sin\theta)+2\cos\theta(1+\sin\theta)}{2\sin\theta(1+\sin\theta)}$
$=\dfrac{2(1+\sin\theta)(1+\cos\theta)}{2\sin\theta(1+\sin\theta)}$
$=\dfrac{1+\cos\theta}{\sin\theta}$

90. $\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}$
$=\dfrac{(1+\cos\theta)+\sin\theta}{(1+\cos\theta)-\sin\theta}\cdot\dfrac{(1+\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$
$=\dfrac{1+2\cos\theta+\cos^2\theta+2\sin\theta(1+\cos\theta)+\sin^2\theta}{1+2\cos\theta+\cos^2\theta-\sin^2\theta}$
$=\dfrac{1+2\cos\theta+\cos^2\theta+2\sin\theta(1+\cos\theta)+1-\cos^2\theta}{1+2\cos\theta+\cos^2\theta-(1-\cos^2\theta)}$
$=\dfrac{2+2\cos\theta+2\sin\theta(1+\cos\theta)}{2\cos\theta+2\cos^2\theta}$
$=\dfrac{2(1+\cos\theta)+2\sin\theta(1+\cos\theta)}{2\cos\theta(1+\cos\theta)}$
$=\dfrac{2(1+\cos\theta)(1+\sin\theta)}{2\cos\theta(1+\cos\theta)}$
$=\dfrac{1+\sin\theta}{\cos\theta}$
$=\dfrac{1}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}$
$=\sec\theta+\tan\theta$

91. $(a\sin\theta+b\cos\theta)^2+(a\cos\theta-b\sin\theta)^2$
$=a^2\sin^2\theta+2ab\sin\theta\cos\theta+b^2\cos^2\theta$
$\quad+a^2\cos^2\theta-2ab\sin\theta\cos\theta+b^2\sin^2\theta$
$=a^2(\sin^2\theta+\cos^2\theta)+b^2(\sin^2\theta+\cos^2\theta)$
$=a^2+b^2$

92. $(2a\sin\theta\cos\theta)^2+a^2(\cos^2\theta-\sin^2\theta)^2$
$=4a^2\sin^2\theta\cos^2\theta$
$\quad+a^2\left(\cos^4\theta-2\cos^2\theta\sin^2\theta+\sin^4\theta\right)$
$=a^2\left(4\sin^2\theta\cos^2\theta+\cos^4\theta-2\cos^2\theta\sin^2\theta+\sin^4\theta\right)$
$=a^2\left(\cos^4\theta+2\cos^2\theta\sin^2\theta+\sin^4\theta\right)$
$=a^2\left(\cos^2\theta+\sin^2\theta\right)^2$
$=a^2(1)^2$
$=a^2$

93. $\dfrac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}=\dfrac{\tan\alpha+\tan\beta}{\dfrac{1}{\tan\alpha}+\dfrac{1}{\tan\beta}}$
$=\dfrac{\tan\alpha+\tan\beta}{\dfrac{\tan\beta+\tan\alpha}{\tan\alpha\tan\beta}}$
$=(\tan\alpha+\tan\beta)\cdot\left(\dfrac{\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}\right)$
$=\tan\alpha\tan\beta$

94. $(\tan\alpha+\tan\beta)(1-\cot\alpha\cot\beta)$
$+(\cot\alpha+\cot\beta)(1-\tan\alpha\tan\beta)$
$=\tan\alpha+\tan\beta-\tan\alpha\cot\alpha\cot\beta$
$-\tan\beta\cot\alpha\cot\beta+\cot\alpha+\cot\beta$
$-\cot\alpha\tan\alpha\tan\beta-\cot\beta\tan\alpha\tan\beta$
$=\tan\alpha+\tan\beta-\cot\beta-\cot\alpha+\cot\alpha$
$+\cot\beta-\tan\beta-\tan\alpha$
$=0$

95. $(\sin\alpha+\cos\beta)^2+(\cos\beta+\sin\alpha)(\cos\beta-\sin\alpha)$
$=\sin^2\alpha+2\sin\alpha\cos\beta+\cos^2\beta+\cos^2\beta-\sin^2\alpha$
$=2\sin\alpha\cos\beta+2\cos^2\beta$
$=2\cos\beta(\sin\alpha+\cos\beta)$

96. $(\sin\alpha-\cos\beta)^2+(\cos\beta+\sin\alpha)(\cos\beta-\sin\alpha)$
$=\sin^2\alpha-2\sin\alpha\cos\beta+\cos^2\beta+\cos^2\beta-\sin^2\alpha$
$=-2\sin\alpha\cos\beta+2\cos^2\beta=-2\cos\beta(\sin\alpha-\cos\beta)$

97. $\ln|\sec\theta|=\ln\left|\dfrac{1}{\cos\theta}\right|=\ln|\cos\theta|^{-1}=-\ln|\cos\theta|$

98. $\ln|\tan\theta|=\ln\left|\dfrac{\sin\theta}{\cos\theta}\right|=\ln|\sin\theta|-\ln|\cos\theta|$

99. $\ln|1+\cos\theta|+\ln|1-\cos\theta|$
$=\ln(|1+\cos\theta|\cdot|1-\cos\theta|)$
$=\ln|1-\cos^2\theta|$
$=\ln|\sin^2\theta|$
$=2\ln|\sin\theta|$

100. $\ln|\sec\theta+\tan\theta|+\ln|\sec\theta-\tan\theta|$
$=\ln(|\sec\theta+\tan\theta|\cdot|\sec\theta-\tan\theta|)$
$=\ln|\sec^2\theta-\tan^2\theta|$
$=\ln|\tan^2\theta+1-\tan^2\theta|$
$=\ln|1|$
$=0$

101. $f(x)=\sin x\cdot\tan x$
$=\sin x\cdot\dfrac{\sin x}{\cos x}$
$=\dfrac{\sin^2 x}{\cos x}$
$=\dfrac{1-\cos^2 x}{\cos x}$
$=\dfrac{1}{\cos x}-\dfrac{\cos^2 x}{\cos x}$
$=\sec x-\cos x$
$=g(x)$

102. $f(x)=\cos x\cdot\cot x$
$=\cos x\cdot\dfrac{\cos x}{\sin x}$
$=\dfrac{\cos^2 x}{\sin x}$
$=\dfrac{1-\sin^2 x}{\sin x}$
$=\dfrac{1}{\sin x}-\dfrac{\sin^2 x}{\sin x}$
$=\csc x-\sin x$
$=g(x)$

103. $f(\theta)=\dfrac{1-\sin\theta}{\cos\theta}-\dfrac{\cos\theta}{1+\sin\theta}$

$$=\frac{(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)}-\frac{\cos\theta\cdot\cos\theta}{(1+\sin\theta)\cdot\cos\theta}$$
$$=\frac{1-\sin^2\theta-\cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$=\frac{1-(\sin^2\theta+\cos^2\theta)}{\cos\theta(1+\sin\theta)}$$
$$=\frac{1-1}{\cos\theta(1+\sin\theta)}$$
$$=\frac{0}{\cos\theta(1+\sin\theta)}$$
$$=0$$
$$=g(\theta)$$

104. $f(\theta)=\tan\theta+\sec\theta$

$$=\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}$$
$$=\frac{1+\sin\theta}{\cos\theta}$$
$$=\frac{1+\sin\theta}{\cos\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}$$
$$=\frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}$$
$$=\frac{\cos^2\theta}{\cos\theta(1-\sin\theta)}$$
$$=\frac{\cos\theta}{1-\sin\theta}$$
$$=g(\theta)$$

105. $\sqrt{16+16\tan^2\theta}=\sqrt{16(1+\tan^2\theta)}=4\sqrt{1+\tan^2\theta}$.

Since $\sec\theta>0$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, then

$4\sqrt{1+\tan^2\theta}=4\sqrt{\sec^2\theta}=4\sec\theta$

106. $\sqrt{9\sec^2\theta-9}=\sqrt{9(\sec^2\theta-1)}=3\sqrt{\sec^2\theta-1}$.

Since $\tan\theta>0$ for $\pi<\theta<\frac{3\pi}{2}$, then

$3\sqrt{\sec^2\theta-1}=3\sqrt{\tan^2\theta}=3\tan\theta$

107. $1200\sec\theta\left(2\sec^2\theta-1\right)=1200\dfrac{1}{\cos\theta}\left(\dfrac{2}{\cos^2\theta}-1\right)$

$$=1200\frac{1}{\cos\theta}\left(\frac{2}{\cos^2\theta}-\frac{\cos^2\theta}{\cos^2\theta}\right)$$
$$=1200\frac{1}{\cos\theta}\left(\frac{2-\cos^2\theta}{\cos^2\theta}\right)$$
$$=\frac{1200\left(1+1-\cos^2\theta\right)}{\cos^3\theta}$$
$$=\frac{1200\left(1+\sin^2\theta\right)}{\cos^3\theta}$$

108. $I_t=4A^2\dfrac{(\csc\theta-1)(\sec\theta+\tan\theta)}{\csc\theta\sec\theta}$

$$=4A^2\frac{\csc\theta-1}{\csc\theta}\cdot\frac{\sec\theta+\tan\theta}{\sec\theta}$$
$$=4A^2\left(1-\frac{1}{\csc\theta}\right)\left(1+\frac{\tan\theta}{\sec\theta}\right)$$
$$=4A^2(1-\sin\theta)(1+\sin\theta)$$
$$=4A^2\left(1-\sin^2\theta\right)$$
$$=4A^2\cos^2\theta=(2A\cos\theta)^2$$

109. Answers will vary.

110. $\sin^2\theta+\cos^2\theta=1$

$\tan^2\theta+1=\sec^2\theta$

$1+\cot^2\theta=\csc^2\theta$

111 – 112. Answers will vary.

113. Since a is negative then the graph opens up so the function has a maximum value. To find the maximum value we can find the vertex.

$$x=-\frac{b}{2a}=-\frac{-120}{2(-3)}=20$$
$$f(20)=-3(20)^2+120(20)+50=1250$$

The vertex is $(20,1250)$ so the maximum value of the function is 1250.

114. $f(x)=\dfrac{x+1}{x-2}; g(x)=3x-4$

$$f\circ g=\frac{(3x-4)+1}{(3x-4)-2}=\frac{3x-3}{3x-6}=\frac{3(x-1)}{3(x-2)}=\frac{x-1}{x-2}$$

115. For the point $(-12,5)$, $x=-12$, $y=5$,

$r=\sqrt{x^2+y^2}=\sqrt{144+25}=\sqrt{169}=13$

$\sin\theta=\dfrac{5}{13}$ $\qquad\csc\theta=\dfrac{13}{5}$

$\cos\theta=-\dfrac{12}{13}$ $\qquad\sec\theta=-\dfrac{13}{12}$

$\tan\theta=-\dfrac{5}{12}$ $\qquad\cot\theta=-\dfrac{12}{5}$

116. $$\frac{f(\pi/2)-f(0)}{\pi/2-0}=\frac{\cos(\pi/2)-\cos(0)}{\pi/2}=\frac{0-1}{\pi/2}=-\frac{2}{\pi}$$

The average rate of change is $-\dfrac{2}{\pi}$.

Section 7.5

1. $\sqrt{(5-2)^2+(1-(-3))^2}$
$=\sqrt{3^2+4^2}=\sqrt{9+16}=\sqrt{25}=5$

2. $-\dfrac{3}{5}$

3. **a.** $\dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2}=\dfrac{\sqrt{2}}{4}$

b. $1-\dfrac{1}{2}=\dfrac{1}{2}$

4. $y=4, r=5, x=-3$ (Quadrant 2)

$\cos\alpha=\dfrac{x}{r}=-\dfrac{3}{5}$

5. $-$

6. $-$

7. False

8. False

9. False

10. True

11. a

12. d

13. $$\cos 165^\circ=\cos(120^\circ+45^\circ)=\cos 120^\circ\cdot\cos 45^\circ-\sin 120^\circ\cdot\sin 45^\circ=-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=-\frac{1}{4}\left(\sqrt{2}+\sqrt{6}\right)$$

14. $$\sin 105^\circ=\sin(60^\circ+45^\circ)=\sin 60^\circ\cdot\cos 45^\circ+\cos 60^\circ\cdot\sin 45^\circ=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

15. $$\tan 15^\circ=\tan(45^\circ-30^\circ)=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\cdot\tan 30^\circ}=\frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}}\cdot\frac{3}{3}=\frac{3-\sqrt{3}}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}}=\frac{9-6\sqrt{3}+3}{9-3}=\frac{12-6\sqrt{3}}{6}=2-\sqrt{3}$$

16. $\tan 195^\circ = \tan(135^\circ + 60^\circ)$

$$= \frac{\tan 135^\circ + \tan 60^\circ}{1 - \tan 135^\circ \cdot \tan 60^\circ}$$

$$= \frac{-1+\sqrt{3}}{1-(-1)\cdot\sqrt{3}}$$

$$= \frac{-1+\sqrt{3}}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{-1+2\sqrt{3}-3}{1-3}$$

$$= \frac{-4+2\sqrt{3}}{-2}$$

$$= 2-\sqrt{3}$$

17. $\sin\frac{5\pi}{12} = \sin\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right)$

$$= \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

18. $\sin\frac{\pi}{12} = \sin\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right)$

$$= \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$$

19. $\cos\frac{7\pi}{12} = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)$

$$= \cos\frac{\pi}{3}\cdot\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$$

20. $\tan\frac{7\pi}{12} = \tan\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{3}}{1 - \tan\frac{\pi}{4}\cdot\tan\frac{\pi}{3}}$$

$$= \frac{1+\sqrt{3}}{1-1\cdot\sqrt{3}}$$

$$= \left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)\cdot\left(\frac{1+\sqrt{3}}{1+\sqrt{3}}\right)$$

$$= \frac{1+2\sqrt{3}+3}{1-3}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

$$= -2-\sqrt{3}$$

21. $\sin\frac{17\pi}{12} = \sin\left(\frac{15\pi}{12} + \frac{2\pi}{12}\right)$

$$= \sin\frac{5\pi}{4}\cdot\cos\frac{\pi}{6} + \cos\frac{5\pi}{4}\cdot\sin\frac{\pi}{6}$$

$$= -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{2}$$

$$= -\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

22. $\tan\frac{19\pi}{12} = \tan\left(\frac{15\pi}{12} + \frac{4\pi}{12}\right)$

$$= \frac{\tan\frac{5\pi}{4} + \tan\frac{\pi}{3}}{1 - \tan\frac{5\pi}{4}\cdot\tan\frac{\pi}{3}}$$

$$= \frac{1+\sqrt{3}}{1-1\cdot\sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{1+2\sqrt{3}+3}{1-3}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

$$= -2-\sqrt{3}$$

23. $\sec\left(-\frac{\pi}{12}\right)=\frac{1}{\cos\left(-\frac{\pi}{12}\right)}=\frac{1}{\cos\left(\frac{3\pi}{12}-\frac{4\pi}{12}\right)}$

$=\frac{1}{\cos\frac{\pi}{4}\cdot\cos\frac{\pi}{3}+\sin\frac{\pi}{4}\cdot\sin\frac{\pi}{3}}$

$=\frac{1}{\frac{\sqrt{2}}{2}\cdot\frac{1}{2}+\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}}$

$=\frac{1}{\frac{\sqrt{2}+\sqrt{6}}{4}}$

$=\frac{4}{\sqrt{2}+\sqrt{6}}\cdot\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}$

$=\frac{4\sqrt{2}-4\sqrt{6}}{2-6}$

$=\frac{4\sqrt{2}-4\sqrt{6}}{-4}$

$=\sqrt{6}-\sqrt{2}$

24. $\cot\left(-\frac{5\pi}{12}\right)=-\cot\frac{5\pi}{12}=\frac{-1}{\tan\frac{5\pi}{12}}$

$=\frac{-1}{\tan\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right)}$

$=\frac{-1}{\frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}}$

$=-\left(\frac{1-\tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}\right)$

$=-\frac{1-1\cdot\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$

$=-\frac{\sqrt{3}-1}{\sqrt{3}+1}\cdot\frac{\sqrt{3}-1}{\sqrt{3}-1}$

$=-\frac{3-\sqrt{3}-\sqrt{3}+1}{3-1}$

$=-\frac{4-2\sqrt{3}}{2}$

$=-2+\sqrt{3}$

25. $\sin 20^\circ\cdot\cos 10^\circ+\cos 20^\circ\cdot\sin 10^\circ=\sin(20^\circ+10^\circ)$

$=\sin 30^\circ$

$=\frac{1}{2}$

26. $\sin 20^\circ\cdot\cos 80^\circ-\cos 20^\circ\cdot\sin 80^\circ=\sin(20^\circ-80^\circ)$

$=\sin(-60^\circ)$

$=-\sin 60^\circ$

$=-\frac{\sqrt{3}}{2}$

27. $\cos 70^\circ\cdot\cos 20^\circ-\sin 70^\circ\cdot\sin 20^\circ=\cos(70^\circ+20^\circ)$

$=\cos 90^\circ$

$=0$

28. $\cos 40^\circ\cdot\cos 10^\circ+\sin 40^\circ\cdot\sin 10^\circ=\cos(40^\circ-10^\circ)$

$=\cos 30^\circ$

$=\frac{\sqrt{3}}{2}$

29. $\frac{\tan 20^\circ+\tan 25^\circ}{1-\tan 20^\circ\tan 25^\circ}=\tan\left(20^\circ+25^\circ\right)$

$=\tan 45^\circ$

$=1$

30. $\frac{\tan 40^\circ-\tan 10^\circ}{1+\tan 40^\circ\tan 10^\circ}=\tan\left(40^\circ-10^\circ\right)$

$=\tan 30^\circ$

$=\frac{\sqrt{3}}{3}$

31. $\sin\frac{\pi}{12}\cdot\cos\frac{7\pi}{12}-\cos\frac{\pi}{12}\cdot\sin\frac{7\pi}{12}=\sin\left(\frac{\pi}{12}-\frac{7\pi}{12}\right)$

$=\sin\left(-\frac{6\pi}{12}\right)$

$=\sin\left(-\frac{\pi}{2}\right)$

$=-1$

32. $\cos\frac{5\pi}{12}\cdot\cos\frac{7\pi}{12}-\sin\frac{5\pi}{12}\cdot\sin\frac{7\pi}{12}=\cos\left(\frac{5\pi}{12}+\frac{7\pi}{12}\right)$

$=\cos\frac{12\pi}{12}$

$=\cos\pi$

$=-1$

33. $\cos\frac{\pi}{12}\cdot\cos\frac{5\pi}{12}+\sin\frac{5\pi}{12}\cdot\sin\frac{\pi}{12}=\cos\left(\frac{\pi}{12}-\frac{5\pi}{12}\right)$

$=\cos\left(-\frac{4\pi}{12}\right)$

$=\cos\left(-\frac{\pi}{3}\right)$

$=\cos\frac{\pi}{3}$

$=\frac{1}{2}$

34. $\sin\frac{\pi}{18}\cdot\cos\frac{5\pi}{18}+\cos\frac{\pi}{18}\cdot\sin\frac{5\pi}{18}=\sin\left(\frac{\pi}{18}+\frac{5\pi}{18}\right)$

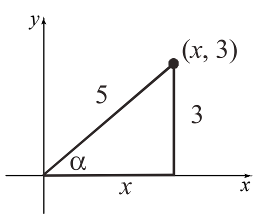
$=\sin\frac{6\pi}{18}$

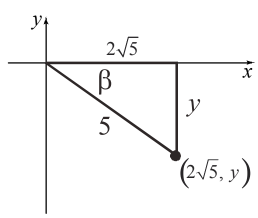
$=\sin\frac{\pi}{3}$

$=\frac{\sqrt{3}}{2}$

35. $\sin\alpha=\frac{3}{5},\ 0<\alpha<\frac{\pi}{2}$

$\cos\beta=\frac{2\sqrt{5}}{5},\ -\frac{\pi}{2}<\beta<0$





$x^2+3^2=5^2,\ x>0$

$x^2=25-9=16,\ x>0$

$x=4$

$\cos\alpha=\frac{4}{5},\qquad \tan\alpha=\frac{3}{4}$

$\left(2\sqrt{5}\right)^2+y^2=5^2,\ y<0$

$y^2=25-20=5,\ y<0$

$y=-\sqrt{5}$

$\sin\beta=-\frac{\sqrt{5}}{5},\quad \tan\beta=\frac{-\sqrt{5}}{2\sqrt{5}}=-\frac{1}{2}$

a. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$=\frac{3}{5}\cdot\frac{2\sqrt{5}}{5}+\frac{4}{5}\cdot\left(-\frac{\sqrt{5}}{5}\right)$

$=\frac{6\sqrt{5}-4\sqrt{5}}{25}$

$=\frac{2\sqrt{5}}{25}$

b. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$=\frac{4}{5}\cdot\frac{2\sqrt{5}}{5}-\frac{3}{5}\cdot\left(-\frac{\sqrt{5}}{5}\right)$

$=\frac{8\sqrt{5}+3\sqrt{5}}{25}$

$=\frac{11\sqrt{5}}{25}$

c. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$=\frac{3}{5}\cdot\frac{2\sqrt{5}}{5}-\frac{4}{5}\cdot\left(-\frac{\sqrt{5}}{5}\right)$

$=\frac{6\sqrt{5}+4\sqrt{5}}{25}$

$=\frac{10\sqrt{5}}{25}$

$=\frac{2\sqrt{5}}{5}$

d. $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$

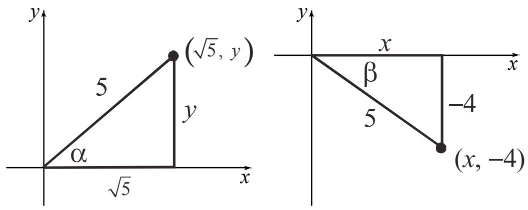
$=\frac{\frac{3}{4}-\left(-\frac{1}{2}\right)}{1+\left(\frac{3}{4}\right)\left(-\frac{1}{2}\right)}$

$=\frac{\frac{5}{4}}{\frac{5}{8}}$

$=2$

36. $\cos\alpha=\frac{\sqrt{5}}{5},\ 0<\alpha<\frac{\pi}{2}$

$\sin\beta=-\frac{4}{5},\ -\frac{\pi}{2}<\beta<0$



$$\left(\sqrt{5}\right)^2 + y^2 = 5^2,\ \ y > 0$$
$$y^2 = 25 - 5 = 20,\ \ y > 0$$
$$y = \sqrt{20} = 2\sqrt{5}$$
$$\sin\alpha = \frac{2\sqrt{5}}{5}, \qquad \tan\alpha = \frac{2\sqrt{5}}{\sqrt{5}} = 2$$

$$x^2 + (-4)^2 = 5^2,\ \ x > 0$$
$$x^2 = 25 - 16 = 9,\ \ x > 0$$
$$x = 3$$
$$\cos\beta = \frac{3}{5}, \qquad \tan\beta = \frac{-4}{3} = -\frac{4}{3}$$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$$= \left(\frac{2\sqrt{5}}{5}\right)\cdot\left(\frac{3}{5}\right) + \left(\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{4}{5}\right)$$
$$= \frac{6\sqrt{5} - 4\sqrt{5}}{25}$$
$$= \frac{2\sqrt{5}}{25}$$

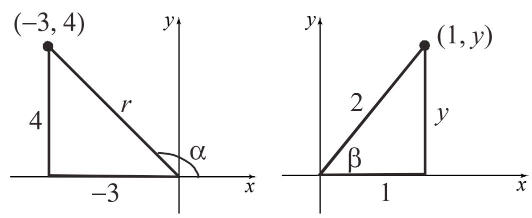
b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$$= \left(\frac{\sqrt{5}}{5}\right)\cdot\left(\frac{3}{5}\right) - \left(\frac{2\sqrt{5}}{5}\right)\cdot\left(-\frac{4}{5}\right)$$
$$= \frac{3\sqrt{5} + 8\sqrt{5}}{25}$$
$$= \frac{11\sqrt{5}}{25}$$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$$= \left(\frac{2\sqrt{5}}{5}\right)\cdot\left(\frac{3}{5}\right) - \left(\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{4}{5}\right)$$
$$= \frac{6\sqrt{5} + 4\sqrt{5}}{25}$$
$$= \frac{10\sqrt{5}}{25}$$
$$= \frac{2\sqrt{5}}{5}$$

d. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$
$$= \frac{2 - \left(-\frac{4}{3}\right)}{1 + 2\cdot\left(-\frac{4}{3}\right)}$$
$$= \frac{\frac{10}{3}}{-\frac{5}{3}}$$
$$= -2$$

37. $\tan\alpha = -\dfrac{4}{3},\ \ \dfrac{\pi}{2} < \alpha < \pi$

$\cos\beta = \dfrac{1}{2},\ \ 0 < \beta < \dfrac{\pi}{2}$



$$r^2 = (-3)^2 + 4^2 = 25$$
$$r = 5$$
$$\sin\alpha = \frac{4}{5}, \qquad \cos\alpha = \frac{-3}{5} = -\frac{3}{5}$$

$$1^2 + y^2 = 2^2,\ \ y > 0$$
$$y^2 = 4 - 1 = 3,\ \ y > 0$$
$$y = \sqrt{3}$$
$$\sin\beta = \frac{\sqrt{3}}{2}, \qquad \tan\beta = \frac{\sqrt{3}}{1} = \sqrt{3}$$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$$= \left(\frac{4}{5}\right)\cdot\left(\frac{1}{2}\right) + \left(-\frac{3}{5}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{4 - 3\sqrt{3}}{10}$$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$$= \left(-\frac{3}{5}\right)\cdot\left(\frac{1}{2}\right) - \left(\frac{4}{5}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{-3 - 4\sqrt{3}}{10}$$

c. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$=\left(\frac{4}{5}\right)\cdot\left(\frac{1}{2}\right)-\left(-\frac{3}{5}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$

$$=\frac{4+3\sqrt{3}}{10}$$

d. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$$=\frac{-\frac{4}{3}-\sqrt{3}}{1+\left(-\frac{4}{3}\right)\cdot\sqrt{3}}$$

$$=\frac{\frac{-4-3\sqrt{3}}{3}}{\frac{3-4\sqrt{3}}{3}}$$

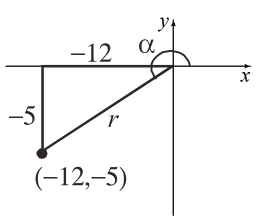
$$=\left(\frac{-4-3\sqrt{3}}{3-4\sqrt{3}}\right)\cdot\left(\frac{3+4\sqrt{3}}{3+4\sqrt{3}}\right)$$

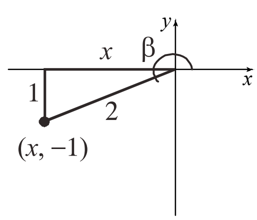
$$=\frac{-48-25\sqrt{3}}{-39}$$

$$=\frac{25\sqrt{3}+48}{39}$$

38. $\tan\alpha=\dfrac{5}{12},\ \pi<\alpha<\dfrac{3\pi}{2}$

$\sin\beta=-\dfrac{1}{2},\ \pi<\beta<\dfrac{3\pi}{2}$





$r^2=(-12)^2+(-5)^2=169$

$r=13$

$\sin\alpha=\dfrac{-5}{13}=-\dfrac{5}{13},\ \cos\alpha=\dfrac{-12}{13}=-\dfrac{12}{13}$

$x^2+(-1)^2=2^2,\ x<0$

$x^2=4-1=3,\ x<0$

$x=-\sqrt{3}$

$\cos\beta=-\dfrac{\sqrt{3}}{2},\qquad \tan\beta=\dfrac{-1}{-\sqrt{3}}=\dfrac{\sqrt{3}}{3}$

a. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\left(-\frac{5}{13}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right)+\left(-\frac{12}{13}\right)\cdot\left(-\frac{1}{2}\right)$$

$$=\frac{5\sqrt{3}+12}{26}=\frac{12+5\sqrt{3}}{26}$$

b. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$=\left(-\frac{12}{13}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right)-\left(-\frac{5}{13}\right)\cdot\left(-\frac{1}{2}\right)$$

$$=\frac{12\sqrt{3}-5}{26}$$

c. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$=\left(-\frac{5}{13}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right)-\left(-\frac{12}{13}\right)\cdot\left(-\frac{1}{2}\right)$$

$$=\frac{5\sqrt{3}-12}{26}$$

d. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$

$$=\frac{\frac{5}{12}-\frac{\sqrt{3}}{3}}{1+\frac{5}{12}\cdot\frac{\sqrt{3}}{3}}=\frac{\frac{5-4\sqrt{3}}{12}}{\frac{36+5\sqrt{3}}{36}}$$

$$=\left(\frac{15-12\sqrt{3}}{36+5\sqrt{3}}\right)\cdot\left(\frac{36-5\sqrt{3}}{36-5\sqrt{3}}\right)$$

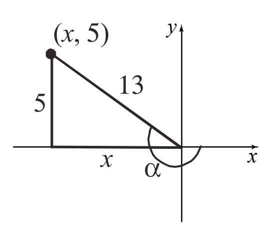
$$=\frac{540-507\sqrt{3}+180}{1296-75}$$

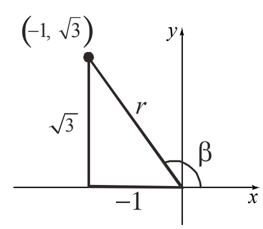
$$=\frac{720-507\sqrt{3}}{1221}$$

$$=\frac{240-169\sqrt{3}}{407}$$

39. $\sin\alpha=\dfrac{5}{13},\ -\dfrac{3\pi}{2}<\alpha<-\pi$

$\tan\beta=-\sqrt{3},\ \dfrac{\pi}{2}<\beta<\pi$





$x^2 + 5^2 = 13^2, \ x < 0$

$x^2 = 169 - 25 = 144, \ x < 0$

$x = -12$

$\cos\alpha = \dfrac{-12}{13} = -\dfrac{12}{13}, \qquad \tan\alpha = -\dfrac{5}{12}$

$r^2 = (-1)^2 + \sqrt{3}^2 = 4$

$r = 2$

$\sin\beta = \dfrac{\sqrt{3}}{2}, \quad \cos\beta = \dfrac{-1}{2} = -\dfrac{1}{2}$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\dfrac{5}{13}\right)\cdot\left(-\dfrac{1}{2}\right) + \left(-\dfrac{12}{13}\right)\cdot\left(\dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{-5-12\sqrt{3}}{26} \text{ or } -\dfrac{5+12\sqrt{3}}{26}$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(-\dfrac{12}{13}\right)\cdot\left(-\dfrac{1}{2}\right) - \left(\dfrac{5}{13}\right)\cdot\left(\dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{12-5\sqrt{3}}{26}$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \left(\dfrac{5}{13}\right)\cdot\left(-\dfrac{1}{2}\right) - \left(-\dfrac{12}{13}\right)\cdot\left(\dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{-5+12\sqrt{3}}{26}$

d. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \dfrac{-\dfrac{5}{12} - \left(-\sqrt{3}\right)}{1 + \left(-\dfrac{5}{12}\right)\cdot\left(-\sqrt{3}\right)}$

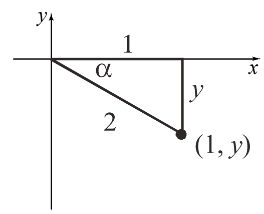
$= \dfrac{\dfrac{-5+12\sqrt{3}}{12}}{\dfrac{12+5\sqrt{3}}{12}}$

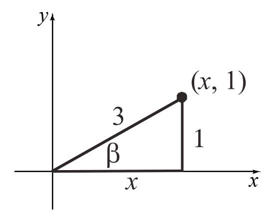
$= \left(\dfrac{-5+12\sqrt{3}}{12+5\sqrt{3}}\right)\cdot\left(\dfrac{12-5\sqrt{3}}{12-5\sqrt{3}}\right)$

$= \dfrac{-240+169\sqrt{3}}{69}$

40. $\cos\alpha = \dfrac{1}{2}, \ -\dfrac{\pi}{2} < \alpha < 0$

$\sin\beta = \dfrac{1}{3}, \ 0 < \beta < \dfrac{\pi}{2}$





$1^2 + y^2 = 2^2, \ y < 0$

$y^2 = 4 - 1 = 3, \ y < 0$

$y = -\sqrt{3}$

$\sin\alpha = \dfrac{-\sqrt{3}}{2} = -\dfrac{\sqrt{3}}{2}, \ \tan\alpha = \dfrac{-\sqrt{3}}{1} = -\sqrt{3}$

$x^2 + 1^2 = 3^2, \ x > 0$

$x^2 = 9 - 1 = 8. \ x > 0$

$x = \sqrt{8} = 2\sqrt{2}$

$\cos\beta = \dfrac{2\sqrt{2}}{3}, \qquad \tan\beta = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(-\dfrac{\sqrt{3}}{2}\right)\cdot\left(\dfrac{2\sqrt{2}}{3}\right) + \left(\dfrac{1}{2}\right)\cdot\left(\dfrac{1}{3}\right)$

$= \dfrac{1-2\sqrt{6}}{6}$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(\dfrac{1}{2}\right)\cdot\left(\dfrac{2\sqrt{2}}{3}\right) - \left(-\dfrac{\sqrt{3}}{2}\right)\cdot\left(\dfrac{1}{3}\right)$

$= \dfrac{\sqrt{3}+2\sqrt{2}}{6}$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \left(-\dfrac{\sqrt{3}}{2}\right)\cdot\left(\dfrac{2\sqrt{2}}{3}\right) - \left(\dfrac{1}{2}\right)\cdot\left(\dfrac{1}{3}\right)$

$= \dfrac{-1-2\sqrt{6}}{6}$

d. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$=\dfrac{-\sqrt{3}-\frac{\sqrt{2}}{4}}{1+\left(-\sqrt{3}\right)\cdot\frac{\sqrt{2}}{4}}$

$=\dfrac{\frac{-4\sqrt{3}-\sqrt{2}}{4}}{\frac{4-\sqrt{6}}{4}}$

$=\left(\dfrac{-4\sqrt{3}-\sqrt{2}}{4-\sqrt{6}}\right)\cdot\left(\dfrac{4+\sqrt{6}}{4+\sqrt{6}}\right)$

$=\dfrac{-16\sqrt{3}-4\sqrt{2}-4\sqrt{18}-\sqrt{12}}{16-6}$

$=\dfrac{-18\sqrt{3}-16\sqrt{2}}{10}$

$=\dfrac{-9\sqrt{3}-8\sqrt{2}}{5}$

d. $\tan\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\cdot\tan\frac{\pi}{4}}$

$=\dfrac{-\frac{1}{2\sqrt{2}}+1}{1-\left(-\frac{1}{2\sqrt{2}}\right)\cdot 1}$

$=\dfrac{\frac{-1+2\sqrt{2}}{2\sqrt{2}}}{\frac{2\sqrt{2}+1}{2\sqrt{2}}}$

$=\left(\dfrac{2\sqrt{2}-1}{2\sqrt{2}+1}\right)\cdot\left(\dfrac{2\sqrt{2}-1}{2\sqrt{2}-1}\right)$

$=\dfrac{8-4\sqrt{2}+1}{8-1}$

$=\dfrac{9-4\sqrt{2}}{7}$

41. $\sin\theta=\dfrac{1}{3},\quad \theta$ in quadrant II

a. $\cos\theta=-\sqrt{1-\sin^2\theta}=-\sqrt{1-\left(\dfrac{1}{3}\right)^2}$

$=-\sqrt{1-\dfrac{1}{9}}$

$=-\sqrt{\dfrac{8}{9}}$

$=-\dfrac{2\sqrt{2}}{3}$

b. $\sin\left(\theta+\dfrac{\pi}{6}\right)=\sin\theta\cdot\cos\dfrac{\pi}{6}+\cos\theta\cdot\sin\dfrac{\pi}{6}$

$=\left(\dfrac{1}{3}\right)\left(\dfrac{\sqrt{3}}{2}\right)+\left(-\dfrac{2\sqrt{2}}{3}\right)\left(\dfrac{1}{2}\right)$

$=\dfrac{\sqrt{3}-2\sqrt{2}}{6}=\dfrac{-2\sqrt{2}+\sqrt{3}}{6}$

c. $\cos\left(\theta-\dfrac{\pi}{3}\right)=\cos\theta\cdot\cos\dfrac{\pi}{3}+\sin\theta\cdot\sin\dfrac{\pi}{3}$

$=\left(-\dfrac{2\sqrt{2}}{3}\right)\left(\dfrac{1}{2}\right)+\left(\dfrac{1}{3}\right)\left(\dfrac{\sqrt{3}}{2}\right)$

$=\dfrac{-2\sqrt{2}+\sqrt{3}}{6}$

42. $\cos\theta=\dfrac{1}{4},\quad \theta$ in quadrant IV

a. $\sin\theta=-\sqrt{1-\cos^2\theta}$

$=-\sqrt{1-\left(\dfrac{1}{4}\right)^2}$

$=-\sqrt{1-\dfrac{1}{16}}$

$=-\sqrt{\dfrac{15}{16}}$

$=-\dfrac{\sqrt{15}}{4}$

b. $\sin\left(\theta-\dfrac{\pi}{6}\right)=\sin\theta\cdot\cos\dfrac{\pi}{6}-\cos\theta\cdot\sin\dfrac{\pi}{6}$

$=\left(-\dfrac{\sqrt{15}}{4}\right)\cdot\left(\dfrac{\sqrt{3}}{2}\right)-\left(\dfrac{1}{4}\right)\cdot\left(\dfrac{1}{2}\right)$

$=\dfrac{-1-3\sqrt{5}}{8}$

c. $\cos\left(\theta+\dfrac{\pi}{3}\right)=\cos\theta\cdot\cos\dfrac{\pi}{3}-\sin\theta\cdot\sin\dfrac{\pi}{3}$

$=\left(\dfrac{1}{4}\right)\cdot\left(\dfrac{1}{2}\right)-\left(-\dfrac{\sqrt{15}}{4}\right)\cdot\left(\dfrac{\sqrt{3}}{2}\right)$

$=\dfrac{1+3\sqrt{5}}{8}$

d. $\tan\left(\theta-\frac{\pi}{4}\right)=\frac{\tan\theta-\tan\frac{\pi}{4}}{1+\tan\theta\cdot\tan\frac{\pi}{4}}$

$$=\frac{-\sqrt{15}-1}{1+\left(-\sqrt{15}\right)\cdot 1}$$

$$=\left(\frac{-1-\sqrt{15}}{1-\sqrt{15}}\right)\cdot\left(\frac{1+\sqrt{15}}{1+\sqrt{15}}\right)$$

$$=\frac{-1-2\sqrt{15}-15}{1-15}$$

$$=\frac{-16-2\sqrt{15}}{-14}$$

$$=\frac{8+\sqrt{15}}{7}$$

43. α lies in quadrant I. Since $x^2+y^2=4$, $r=\sqrt{4}=2$. Now, $(x,1)$ is on the circle, so

$$x^2+1^2=4$$
$$x^2=4-1^2$$
$$x=\sqrt{4-1^2}=\sqrt{3}$$

Thus, $\sin\alpha=\frac{y}{r}=\frac{1}{2}$ and $\cos\alpha=\frac{x}{r}=\frac{\sqrt{3}}{2}$.

β lies in quadrant IV. Since $x^2+y^2=1$, $r=\sqrt{1}=1$. Now, $\left(\frac{1}{3},y\right)$ is on the circle, so

$$\left(\frac{1}{3}\right)^2+y^2=1$$
$$y^2=1-\left(\frac{1}{3}\right)^2$$
$$y=-\sqrt{1-\left(\frac{1}{3}\right)^2}=-\sqrt{\frac{8}{9}}=-\frac{2\sqrt{2}}{3}$$

Thus, $\sin\beta=\frac{y}{r}=\frac{\frac{-2\sqrt{2}}{3}}{1}=\frac{-2\sqrt{2}}{3}$ and $\cos\beta=\frac{x}{r}=\frac{\frac{1}{3}}{1}=\frac{1}{3}$. Thus,

$$f(\alpha+\beta)=\sin(\alpha+\beta)$$
$$=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta$$
$$=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$=\frac{1}{6}-\frac{2\sqrt{6}}{6}=\frac{1-2\sqrt{6}}{6}$$

44. From the solution to Problem 43, we have $\sin\alpha=\frac{1}{2}$, $\cos\alpha=\frac{\sqrt{3}}{2}$, $\sin\beta=\frac{-2\sqrt{2}}{3}$, and $\cos\beta=\frac{1}{3}$. Thus,

$$g(\alpha+\beta)=\cos(\alpha+\beta)$$
$$=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta$$
$$=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right)-\left(\frac{1}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$=\frac{\sqrt{3}}{6}+\frac{2\sqrt{2}}{6}=\frac{\sqrt{3}+2\sqrt{2}}{6}$$

45. From the solution to Problem 43, we have $\sin\alpha=\frac{1}{2}$, $\cos\alpha=\frac{\sqrt{3}}{2}$, $\sin\beta=\frac{-2\sqrt{2}}{3}$, and $\cos\beta=\frac{1}{3}$. Thus,

$$g(\alpha-\beta)=\cos(\alpha-\beta)$$
$$=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta$$
$$=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right)+\left(\frac{1}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$=\frac{\sqrt{3}}{6}-\frac{2\sqrt{2}}{6}=\frac{\sqrt{3}-2\sqrt{2}}{6}$$

46. From the solution to Problem 43, we have $\sin\alpha=\frac{1}{2}$, $\cos\alpha=\frac{\sqrt{3}}{2}$, $\sin\beta=\frac{-2\sqrt{2}}{3}$, and $\cos\beta=\frac{1}{3}$. Thus,

$$f(\alpha-\beta)=\sin(\alpha-\beta)$$
$$=\sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta$$
$$=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$=\frac{1}{6}+\frac{2\sqrt{6}}{6}=\frac{1+2\sqrt{6}}{6}$$

47. From the solution to Problem 43, we have $\sin\alpha=\frac{1}{2}$, $\cos\alpha=\frac{\sqrt{3}}{2}$, $\sin\beta=\frac{-2\sqrt{2}}{3}$, and $\cos\beta=\frac{1}{3}$. Thus,

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \text{ and}$$

$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{-\frac{2\sqrt{2}}{3}}{\frac{1}{3}} = -2\sqrt{2}. \text{ Finally,}$$

$$\begin{aligned} h(\alpha+\beta) = \tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \\ &= \frac{\frac{\sqrt{3}}{3} + \left(-2\sqrt{2}\right)}{1 - \frac{\sqrt{3}}{3}\left(-2\sqrt{2}\right)} \\ &= \frac{\frac{\sqrt{3}}{3} - 2\sqrt{2}}{1+\frac{2\sqrt{6}}{3}} \cdot \frac{3}{3} \\ &= \frac{\sqrt{3}-6\sqrt{2}}{3+2\sqrt{6}} \cdot \frac{3-2\sqrt{6}}{3-2\sqrt{6}} \\ &= \frac{3\sqrt{3}-6\sqrt{2}-18\sqrt{2}+24\sqrt{3}}{9-6\sqrt{6}+6\sqrt{6}-24} \\ &= \frac{27\sqrt{3}-24\sqrt{2}}{-15} = \frac{8\sqrt{2}-9\sqrt{3}}{5} \end{aligned}$$

48. From the solution to Problem 47, we have $\tan\alpha = \frac{\sqrt{3}}{3}$ and $\tan\beta = -2\sqrt{2}$. Thus,

$$\begin{aligned} h(\alpha-\beta) = \tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} \\ &= \frac{\frac{\sqrt{3}}{3} - \left(-2\sqrt{2}\right)}{1 + \frac{\sqrt{3}}{3}\left(-2\sqrt{2}\right)} \\ &= \frac{\frac{\sqrt{3}}{3} + 2\sqrt{2}}{1-\frac{2\sqrt{6}}{3}} \cdot \frac{3}{3} \\ &= \frac{\sqrt{3}+6\sqrt{2}}{3-2\sqrt{6}} \cdot \frac{3+2\sqrt{6}}{3+2\sqrt{6}} \\ &= \frac{3\sqrt{3}+6\sqrt{2}+18\sqrt{2}+24\sqrt{3}}{9+6\sqrt{6}-6\sqrt{6}-24} \\ &= \frac{27\sqrt{3}+24\sqrt{2}}{-15} = -\frac{8\sqrt{2}+9\sqrt{3}}{5} \end{aligned}$$

49. $$\begin{aligned} \sin\left(\frac{\pi}{2}+\theta\right) &= \sin\frac{\pi}{2}\cdot\cos\theta + \cos\frac{\pi}{2}\cdot\sin\theta \\ &= 1\cdot\cos\theta + 0\cdot\sin\theta \\ &= \cos\theta \end{aligned}$$

50. $$\begin{aligned} \cos\left(\frac{\pi}{2}+\theta\right) &= \cos\frac{\pi}{2}\cdot\cos\theta - \sin\frac{\pi}{2}\cdot\sin\theta \\ &= 0\cdot\cos\theta - 1\cdot\sin\theta \\ &= -\sin\theta \end{aligned}$$

51. $$\begin{aligned} \sin(\pi-\theta) &= \sin\pi\cdot\cos\theta - \cos\pi\cdot\sin\theta \\ &= 0\cdot\cos\theta - (-1)\sin\theta \\ &= \sin\theta \end{aligned}$$

52. $$\begin{aligned} \cos(\pi-\theta) &= \cos\pi\cdot\cos\theta + \sin\pi\cdot\sin\theta \\ &= -1\cdot\cos\theta + 0\cdot\sin\theta \\ &= -\cos\theta \end{aligned}$$

53. $$\begin{aligned} \sin(\pi+\theta) &= \sin\pi\cdot\cos\theta + \cos\pi\cdot\sin\theta \\ &= 0\cdot\cos\theta + (-1)\sin\theta \\ &= -\sin\theta \end{aligned}$$

54. $$\begin{aligned} \cos(\pi+\theta) &= \cos\pi\cdot\cos\theta - \sin\pi\cdot\sin\theta \\ &= -1\cdot\cos\theta - 0\cdot\sin\theta \\ &= -\cos\theta \end{aligned}$$

55. $$\begin{aligned} \tan(\pi-\theta) &= \frac{\tan\pi - \tan\theta}{1+\tan\pi\cdot\tan\theta} \\ &= \frac{0-\tan\theta}{1+0\cdot\tan\theta} \\ &= \frac{-\tan\theta}{1} \\ &= -\tan\theta \end{aligned}$$

56. $$\begin{aligned} \tan(2\pi-\theta) &= \frac{\tan 2\pi - \tan\theta}{1+\tan 2\pi\cdot\tan\theta} \\ &= \frac{0-\tan\theta}{1+0\cdot\tan\theta} \\ &= \frac{-\tan\theta}{1} \\ &= -\tan\theta \end{aligned}$$

57. $$\begin{aligned} \sin\left(\frac{3\pi}{2}+\theta\right) &= \sin\frac{3\pi}{2}\cdot\cos\theta + \cos\frac{3\pi}{2}\cdot\sin\theta \\ &= -1\cdot\cos\theta + 0\cdot\sin\theta \\ &= -\cos\theta \end{aligned}$$

58. $\cos\left(\frac{3\pi}{2}+\theta\right)=\cos\frac{3\pi}{2}\cdot\cos\theta-\sin\frac{3\pi}{2}\cdot\sin\theta$
$$=0\cdot\cos\theta-(-1)\cdot\sin\theta$$
$$=\sin\theta$$

59. $\sin(\alpha+\beta)+\sin(\alpha-\beta)$
$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$=2\sin\alpha\cos\beta$$

60. $\cos(\alpha+\beta)+\cos(\alpha-\beta)$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$=2\cos\alpha\cos\beta$$

61. $\frac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta}$
$$=\frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}$$
$$=1+\cot\alpha\tan\beta$$

62. $\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}$
$$=\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$
$$=\tan\alpha+\tan\beta$$

63. $\frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta}=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$
$$=\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$$
$$=1-\tan\alpha\tan\beta$$

64. $\frac{\cos(\alpha-\beta)}{\sin\alpha\cos\beta}=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta}$
$$=\frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta}$$
$$=\cot\alpha+\tan\beta$$

65. $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$
$$=\frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$=\frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$=\frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$$

66. $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$
$$=\frac{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$=\frac{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$=\frac{1-\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta}$$

67. $\cot(\alpha+\beta)=\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}$
$$=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$
$$=\frac{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$
$$=\frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\frac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}+\frac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$
$$=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$$

68. $\cot(\alpha-\beta)=\dfrac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)}$

$$=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

$$=\frac{\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$=\frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}+\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$$

69. $\sec(\alpha+\beta)=\dfrac{1}{\cos(\alpha+\beta)}$

$$=\frac{1}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$=\frac{\dfrac{1}{\sin\alpha\sin\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$=\frac{\dfrac{1}{\sin\alpha}\cdot\dfrac{1}{\sin\beta}}{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$=\frac{\csc\alpha\csc\beta}{\cot\alpha\cot\beta-1}$$

70. $\sec(\alpha-\beta)=\dfrac{1}{\cos(\alpha-\beta)}$

$$=\frac{1}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

$$=\frac{\dfrac{1}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\dfrac{1}{\cos\alpha}\cdot\dfrac{1}{\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\sec\alpha\sec\beta}{1+\tan\alpha\tan\beta}$$

71. $\sin(\alpha-\beta)\sin(\alpha+\beta)$

$$=\left(\sin\alpha\cos\beta-\cos\alpha\sin\beta\right)\left(\sin\alpha\cos\beta+\cos\alpha\sin\beta\right)$$
$$=\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta$$
$$=\sin^2\alpha(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta$$
$$=\sin^2\alpha-\sin^2\alpha\sin^2\beta-\sin^2\beta+\sin^2\alpha\sin^2\beta$$
$$=\sin^2\alpha-\sin^2\beta$$

72. $\cos(\alpha-\beta)\cos(\alpha+\beta)$

$$=\left(\cos\alpha\cos\beta+\sin\alpha\sin\beta\right)\left(\cos\alpha\cos\beta-\sin\alpha\sin\beta\right)$$
$$=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta$$
$$=\cos^2\alpha(1-\sin^2\beta)-(1-\cos^2\alpha)\sin^2\beta$$
$$=\cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta$$
$$=\cos^2\alpha-\sin^2\beta$$

73. $\sin(\theta+k\pi)=\sin\theta\cdot\cos k\pi+\cos\theta\cdot\sin k\pi$

$$=(\sin\theta)(-1)^k+(\cos\theta)(0)$$
$$=(-1)^k\sin\theta,\quad k \text{ any integer}$$

74. $\cos(\theta+k\pi)=\cos\theta\cdot\cos k\pi-\sin\theta\cdot\sin k\pi$

$$=(\cos\theta)(-1)^k-(\sin\theta)(0)$$
$$=(-1)^k\cos\theta,\quad k \text{ any integer}$$

75. $\sin\left(\sin^{-1}\dfrac{1}{2}+\cos^{-1}0\right)=\sin\left(\dfrac{\pi}{6}+\dfrac{\pi}{2}\right)$

$$=\sin\left(\frac{2\pi}{3}\right)$$
$$=\frac{\sqrt{3}}{2}$$

76. $\sin\left(\sin^{-1}\dfrac{\sqrt{3}}{2}+\cos^{-1}1\right)=\sin\left(\dfrac{\pi}{3}+0\right)$

$$=\sin\frac{\pi}{3}$$
$$=\frac{\sqrt{3}}{2}$$

77. $\sin\left[\sin^{-1}\dfrac{3}{5}-\cos^{-1}\left(-\dfrac{4}{5}\right)\right]$

Let $\alpha=\sin^{-1}\dfrac{3}{5}$ and $\beta=\cos^{-1}\left(-\dfrac{4}{5}\right)$. α is in quadrant I; β is in quadrant II. Then $\sin\alpha = \frac{3}{5}$, $0 \le \alpha \le \frac{\pi}{2}$, and $\cos\beta = -\frac{4}{5}$, $\frac{\pi}{2} \le \beta \le \pi$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(-\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sin\left[\sin^{-1}\frac{3}{5} - \cos^{-1}\left(-\frac{4}{5}\right)\right] = \sin(\alpha-\beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$= \left(\frac{3}{5}\right)\cdot\left(-\frac{4}{5}\right) - \left(\frac{4}{5}\right)\cdot\left(\frac{3}{5}\right)$$
$$= -\frac{12}{25} - \frac{12}{25}$$
$$= -\frac{24}{25}$$

78. $\sin\left[\sin^{-1}\left(-\frac{4}{5}\right) - \tan^{-1}\frac{3}{4}\right]$

Let $\alpha = \sin^{-1}\left(-\frac{4}{5}\right)$ and $\beta = \tan^{-1}\frac{3}{4}$. α is in quadrant IV; β is in quadrant I. Then $\sin\alpha = -\frac{4}{5}$, $-\frac{\pi}{2} \le \alpha \le 0$, and $\tan\beta = \frac{3}{4}$, $0 < \beta < \frac{\pi}{2}$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(-\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sec\beta = \sqrt{1+\tan^2\beta} = \sqrt{1+\left(\frac{3}{4}\right)^2} = \sqrt{1-\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

$$\cos\beta = \frac{4}{5}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sin\left[\sin^{-1}\left(-\frac{4}{5}\right) - \tan^{-1}\frac{3}{4}\right]$$
$$= \sin(\alpha-\beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$= \left(-\frac{4}{5}\right)\cdot\left(\frac{4}{5}\right) - \left(\frac{3}{5}\right)\cdot\left(\frac{3}{5}\right)$$
$$= -\frac{16}{25} - \frac{9}{25} = -\frac{25}{25}$$
$$= -1$$

79. $\cos\left(\tan^{-1}\frac{4}{3} + \cos^{-1}\frac{5}{13}\right)$

Let $\alpha = \tan^{-1}\frac{4}{3}$ and $\beta = \cos^{-1}\frac{5}{13}$. α is in quadrant I; β is in quadrant I. Then $\tan\alpha = \frac{4}{3}$, $0 < \alpha < \frac{\pi}{2}$, and $\cos\beta = \frac{5}{13}$, $0 \le \beta \le \frac{\pi}{2}$.

$$\sec\alpha = \sqrt{1+\tan^2\alpha} = \sqrt{1+\left(\frac{4}{3}\right)^2} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$$

$$\cos\alpha = \frac{3}{5}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

$$\cos\left(\tan^{-1}\frac{4}{3}+\cos^{-1}\frac{5}{13}\right)$$
$$=\cos(\alpha+\beta)$$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\left(\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\cdot\left(\frac{12}{13}\right)$$
$$=\frac{15}{65}-\frac{48}{65}=-\frac{33}{65}$$

80. $\cos\left[\tan^{-1}\frac{5}{12}-\sin^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha=\tan^{-1}\frac{5}{12}$ and $\beta=\sin^{-1}\left(-\frac{3}{5}\right)$. α is in quadrant I; β is in quadrant IV. Then $\tan\alpha=\frac{5}{12}$, $0<\alpha<\frac{\pi}{2}$, and $\sin\beta=-\frac{3}{5}$, $-\frac{\pi}{2}<\alpha<0$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}$$
$$=\sqrt{1+\left(\frac{5}{12}\right)^2}=\sqrt{1+\frac{25}{144}}=\sqrt{\frac{169}{144}}=\frac{13}{12}$$
$$\cos\alpha=\frac{12}{13}$$
$$\sin\alpha=\sqrt{1-\cos^2\alpha}$$
$$=\sqrt{1-\left(\frac{12}{13}\right)^2}=\sqrt{1-\frac{144}{169}}=\sqrt{\frac{25}{169}}=\frac{5}{13}$$
$$\cos\beta=\sqrt{1-\sin^2\beta}$$
$$=\sqrt{1-\left(-\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$
$$\cos\left[\tan^{-1}\frac{5}{12}-\sin^{-1}\left(-\frac{3}{5}\right)\right]$$
$$=\cos(\alpha-\beta)$$
$$=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$=\left(\frac{12}{13}\right)\cdot\left(\frac{4}{5}\right)+\left(\frac{5}{13}\right)\cdot\left(-\frac{3}{5}\right)=\frac{48}{65}-\frac{15}{65}=\frac{33}{65}$$

81. $\cos\left(\sin^{-1}\frac{5}{13}-\tan^{-1}\frac{3}{4}\right)$

Let $\alpha=\sin^{-1}\frac{5}{13}$ and $\beta=\tan^{-1}\frac{3}{4}$. α is in quadrant I; β is in quadrant I. Then $\sin\alpha=\frac{5}{13}$, $0\le\alpha\le\frac{\pi}{2}$, and $\tan\beta=\frac{3}{4}$, $0<\beta<\frac{\pi}{2}$.

$$\cos\alpha=\sqrt{1-\sin^2\alpha}$$
$$=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{169}}=\sqrt{\frac{144}{169}}=\frac{12}{13}$$
$$\sec\beta=\sqrt{1+\tan^2\beta}$$
$$=\sqrt{1+\left(\frac{3}{4}\right)^2}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$$
$$\cos\beta=\frac{4}{5}$$
$$\sin\beta=\sqrt{1-\cos^2\beta}$$
$$=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$
$$\cos\left[\sin^{-1}\frac{5}{13}-\tan^{-1}\frac{3}{4}\right]$$
$$=\cos(\alpha-\beta)$$
$$=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$=\frac{12}{13}\cdot\frac{4}{5}+\frac{5}{13}\cdot\frac{3}{5}$$
$$=\frac{48}{65}+\frac{15}{65}$$
$$=\frac{63}{65}$$

82. $\cos\left(\tan^{-1}\frac{4}{3}+\cos^{-1}\frac{12}{13}\right)$

Let $\alpha=\tan^{-1}\frac{4}{3}$ and $\beta=\cos^{-1}\frac{12}{13}$. α is in quadrant I; β is in quadrant I. Then $\tan\alpha=\frac{4}{3}$, $0<\alpha<\frac{\pi}{2}$, and $\cos\beta=\frac{12}{13}$, $0\le\beta\le\frac{\pi}{2}$.

$\sec\alpha = \sqrt{1+\tan^2\alpha}$

$= \sqrt{1+\left(\frac{4}{3}\right)^2} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$

$\cos\alpha = \frac{3}{5}$

$\sin\alpha = \sqrt{1-\cos^2\alpha}$

$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\sin\beta = \sqrt{1-\cos^2\beta}$

$= \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$

$\cos\left(\tan^{-1}\frac{4}{3}+\cos^{-1}\frac{12}{13}\right)$

$= \cos(\alpha+\beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(\frac{3}{5}\right)\cdot\left(\frac{12}{13}\right) - \left(\frac{4}{5}\right)\cdot\left(\frac{5}{13}\right)$

$= \frac{36}{65} - \frac{20}{65}$

$= \frac{16}{65}$

83. $\tan\left(\sin^{-1}\frac{3}{5}+\frac{\pi}{6}\right)$

Let $\alpha = \sin^{-1}\frac{3}{5}$. α is in quadrant I. Then

$\sin\alpha = \frac{3}{5},\ 0 \le \alpha \le \frac{\pi}{2}$.

$\cos\alpha = \sqrt{1-\sin^2\alpha}$

$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{5}\cdot\frac{5}{4} = \frac{3}{4}$

$\tan\left(\sin^{-1}\frac{3}{5}+\frac{\pi}{6}\right) = \frac{\tan\left(\sin^{-1}\frac{3}{5}\right)+\tan\frac{\pi}{6}}{1-\tan\left(\sin^{-1}\frac{3}{5}\right)\cdot\tan\frac{\pi}{6}}$

$= \frac{\frac{3}{4}+\frac{\sqrt{3}}{3}}{1-\frac{3}{4}\cdot\frac{\sqrt{3}}{3}}$

$= \frac{\frac{9+\sqrt{3}}{12}}{\frac{12-3\sqrt{3}}{12}}$

$= \frac{9+\sqrt{3}}{12-3\sqrt{3}}\cdot\frac{12+3\sqrt{3}}{12+3\sqrt{3}}$

$= \frac{108+75\sqrt{3}+36}{144-27}$

$= \frac{144+75\sqrt{3}}{117}$

$= \frac{48+25\sqrt{3}}{39}$

84. $\tan\left(\frac{\pi}{4}-\cos^{-1}\frac{3}{5}\right)$

Let $\alpha = \cos^{-1}\frac{3}{5}$. α is in quadrant I. Then

$\cos\alpha = \frac{3}{5},\ 0 \le \alpha \le \frac{\pi}{2}$.

$\sin\alpha = \sqrt{1-\cos^2\alpha}$

$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{5}\cdot\frac{5}{3} = \frac{4}{3}$

$\tan\left(\frac{\pi}{4}-\cos^{-1}\frac{3}{5}\right) = \frac{\tan\frac{\pi}{4}-\tan\left(\cos^{-1}\frac{3}{5}\right)}{1+\tan\frac{\pi}{4}\cdot\tan\left(\cos^{-1}\frac{3}{5}\right)}$

$= \frac{1-\frac{4}{3}}{1+1\cdot\frac{4}{3}} = \frac{-\frac{1}{3}}{\frac{7}{3}} = -\frac{1}{3}\cdot\frac{3}{7} = -\frac{1}{7}$

85. $\tan\left(\sin^{-1}\frac{4}{5}+\cos^{-1}1\right)$

Let $\alpha=\sin^{-1}\frac{4}{5}$ and $\beta=\cos^{-1}1$; α is in quadrant I. Then $\sin\alpha=\frac{4}{5}$, $0\le\alpha\le\frac{\pi}{2}$, and $\cos\beta=1$, $0\le\beta\le\pi$. So, $\beta=\cos^{-1}1=0$.

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\frac{4}{5}}{\frac{3}{5}}=\frac{4}{5}\cdot\frac{5}{3}=\frac{4}{3}$$

$$\tan\left(\sin^{-1}\frac{4}{5}-\cos^{-1}1\right)=\frac{\tan\left(\sin^{-1}\frac{4}{5}\right)+\tan\left(\cos^{-1}1\right)}{1-\tan\left(\sin^{-1}\frac{4}{5}\right)\cdot\tan\left(\cos^{-1}1\right)}=\frac{\frac{4}{3}+0}{1-\frac{4}{3}\cdot 0}=\frac{\frac{4}{3}}{1}=\frac{4}{3}$$

86. $\tan\left(\cos^{-1}\frac{4}{5}+\sin^{-1}1\right)$

Let $\alpha=\cos^{-1}\frac{4}{5}$ and $\beta=\sin^{-1}1$; α is in quadrant I. Then $\cos\alpha=\frac{4}{5}$, $0\le\alpha\le\frac{\pi}{2}$, and $\sin\beta=1$, $-\frac{\pi}{2}\le\beta\le\frac{\pi}{2}$. So, $\beta=\sin^{-1}1=\frac{\pi}{2}$.

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\frac{3}{5}}{\frac{4}{5}}=\frac{3}{5}\cdot\frac{5}{4}=\frac{3}{4}$, but $\tan\frac{\pi}{2}$ is undefined. Therefore, we cannot use the sum formula for tangent. Rewriting using sine and cosine, we obtain:

$$\tan\left(\cos^{-1}\frac{4}{5}+\sin^{-1}1\right)=\frac{\sin\left(\cos^{-1}\frac{4}{5}+\sin^{-1}1\right)}{\tan\left(\cos^{-1}\frac{4}{5}+\sin^{-1}1\right)}=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}=\frac{\left(\frac{3}{5}\right)(0)+\left(\frac{4}{5}\right)(1)}{\left(\frac{4}{5}\right)(0)-\left(\frac{3}{5}\right)(1)}=\frac{\frac{4}{5}}{-\frac{3}{5}}=-\frac{4}{3}$$

87. $\cos\left(\cos^{-1}u+\sin^{-1}v\right)$

Let $\alpha=\cos^{-1}u$ and $\beta=\sin^{-1}v$.
Then $\cos\alpha=u,\ 0\le\alpha\le\pi$, and

$\sin\beta=v,\ -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}$

$-1\le u\le1,\ -1\le v\le1$

$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-u^2}$

$\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-v^2}$

$$\cos\left(\cos^{-1}u+\sin^{-1}v\right)=\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=u\sqrt{1-v^2}-v\sqrt{1-u^2}$$

88. $\sin\left(\sin^{-1}u-\cos^{-1}v\right)$

Let $\alpha=\sin^{-1}u$ and $\beta=\cos^{-1}v$. Then

$\sin\alpha=u,\ -\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$, and

$\cos\beta=v,\ 0\le\beta\le\pi$.

$-1\le u\le1,\ -1\le v\le1$

$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-u^2}$

$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-v^2}$

$$\sin\left(\sin^{-1}u-\cos^{-1}v\right)=\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta=uv-\sqrt{1-u^2}\sqrt{1-v^2}$$

89. $\sin\left(\tan^{-1} u - \sin^{-1} v\right)$

Let $\alpha = \tan^{-1} u$ and $\beta = \sin^{-1} v$. Then

$\tan\alpha = u,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and

$\sin\beta = v, -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$.

$-\infty < u < \infty$, $-1 \le v \le 1$

$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{u^2 + 1}$

$\cos\alpha = \frac{1}{\sqrt{u^2+1}}$

$\cos\beta = \sqrt{1 - \sin^2\beta} = \sqrt{1 - v^2}$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1 - \frac{1}{u^2+1}} = \sqrt{\frac{u^2+1-1}{u^2+1}} = \sqrt{\frac{u^2}{u^2+1}} = \frac{u}{\sqrt{u^2+1}}$$

$$\sin\left(\tan^{-1} u - \sin^{-1} v\right)$$
$$= \sin(\alpha - \beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$= \frac{u}{\sqrt{u^2+1}} \cdot \sqrt{1-v^2} - \frac{1}{\sqrt{u^2+1}} \cdot v$$
$$= \frac{u\sqrt{1-v^2} - v}{\sqrt{u^2+1}}$$

90. $\cos\left(\tan^{-1} u + \tan^{-1} v\right)$

Let $\alpha = \tan^{-1} u$ and $\beta = \tan^{-1} v$. Then

$\tan\alpha = u,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and

$\tan\beta = v, -\frac{\pi}{2} < \beta < \frac{\pi}{2}$.

$-\infty < u < \infty$, $-\infty < v < \infty$

$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{u^2 + 1}$

$\cos\alpha = \frac{1}{\sqrt{u^2+1}}$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1 - \frac{1}{u^2+1}} = \sqrt{\frac{u^2+1-1}{u^2+1}} = \sqrt{\frac{u^2}{u^2+1}} = \frac{u}{\sqrt{u^2+1}}$$

$\sec\beta = \sqrt{\tan^2\beta + 1} = \sqrt{v^2 + 1}$

$\cos\beta = \frac{1}{\sqrt{v^2+1}}$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1 - \frac{1}{v^2+1}} = \sqrt{\frac{v^2+1-1}{v^2+1}} = \sqrt{\frac{v^2}{v^2+1}} = \frac{v}{\sqrt{v^2+1}}$$

$$\cos\left(\tan^{-1} u + \tan^{-1} v\right)$$
$$= \cos(\alpha + \beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= \frac{1}{\sqrt{u^2+1}} \cdot \frac{1}{\sqrt{v^2+1}} - \frac{u}{\sqrt{u^2+1}} \cdot \frac{v}{\sqrt{v^2+1}}$$
$$= \frac{1 - uv}{\sqrt{u^2+1} \cdot \sqrt{v^2+1}}$$

91. $\tan\left(\sin^{-1} u - \cos^{-1} v\right)$

Let $\alpha = \sin^{-1} u$ and $\beta = \cos^{-1} v$. Then

$\sin\alpha = u,\ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and

$\cos\beta = v, 0 \le \beta \le \pi$.

$-1 \le u \le 1$, $-1 \le v \le 1$

$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - u^2}$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{u}{\sqrt{1-u^2}}$

$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-v^2}$

$\tan\beta = \dfrac{\sin\beta}{\cos\beta} = \dfrac{\sqrt{1-v^2}}{v}$

$$\begin{aligned}\tan\left(\sin^{-1}u - \cos^{-1}v\right) &= \tan(\alpha-\beta)\\ &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\\ &= \frac{\dfrac{u}{\sqrt{1-u^2}} - \dfrac{\sqrt{1-v^2}}{v}}{1+\dfrac{u}{\sqrt{1-u^2}}\cdot\dfrac{\sqrt{1-v^2}}{v}}\\ &= \frac{\dfrac{uv-\sqrt{1-u^2}\sqrt{1-v^2}}{v\sqrt{1-u^2}}}{\dfrac{v\sqrt{1-u^2}+u\sqrt{1-v^2}}{v\sqrt{1-u^2}}}\\ &= \frac{uv-\sqrt{1-u^2}\sqrt{1-v^2}}{v\sqrt{1-u^2}+u\sqrt{1-v^2}}\end{aligned}$$

92. $\sec\left(\tan^{-1}u + \cos^{-1}v\right)$

Let $\alpha = \tan^{-1}u$ and $\beta = \cos^{-1}v$. Then

$\tan\alpha = u,\ -\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}$, and

$\cos\beta = v,\ 0 \le \beta \le \pi$.

$-\infty < u < \infty,\ -1 \le v \le 1$

$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{u^2+1}$

$\cos\alpha = \dfrac{1}{\sqrt{u^2+1}}$

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ &= \sqrt{1-\frac{1}{u^2+1}}\\ &= \sqrt{\frac{u^2+1-1}{u^2+1}}\\ &= \sqrt{\frac{u^2}{u^2+1}}\\ &= \frac{u}{\sqrt{u^2+1}}\end{aligned}$$

$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-v^2}$

$$\begin{aligned}&\sec\left(\tan^{-1}u+\cos^{-1}v\right)\\ &= \sec(\alpha+\beta)\\ &= \frac{1}{\cos(\alpha+\beta)}\\ &= \frac{1}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\\ &= \frac{1}{\dfrac{1}{\sqrt{u^2+1}}\cdot v - \dfrac{u}{\sqrt{u^2+1}}\cdot\sqrt{1-v^2}}\\ &= \frac{1}{\dfrac{v}{\sqrt{u^2+1}} - \dfrac{u\sqrt{1-v^2}}{\sqrt{u^2+1}}}\\ &= \frac{1}{\dfrac{v-u\sqrt{1-v^2}}{\sqrt{u^2+1}}}\\ &= \frac{\sqrt{u^2+1}}{v-u\sqrt{1-v^2}}\end{aligned}$$

93. $\sin\theta - \sqrt{3}\cos\theta = 1$

Divide each side by 2:

$\dfrac{1}{2}\sin\theta - \dfrac{\sqrt{3}}{2}\cos\theta = \dfrac{1}{2}$

Rewrite in the difference of two angles form using $\cos\phi = \dfrac{1}{2}$, $\sin\phi = \dfrac{\sqrt{3}}{2}$, and $\phi = \dfrac{\pi}{3}$:

$$\sin\theta\cos\phi - \cos\theta\sin\phi = \frac{1}{2}$$

$$\sin(\theta-\phi) = \frac{1}{2}$$

$$\theta-\phi = \frac{\pi}{6} \quad\text{or}\quad \theta-\phi = \frac{5\pi}{6}$$

$$\theta - \frac{\pi}{3} = \frac{\pi}{6} \qquad \theta-\frac{\pi}{3} = \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{2} \qquad \theta = \frac{7\pi}{6}$$

The solution set is $\left\{\dfrac{\pi}{2},\ \dfrac{7\pi}{6}\right\}$.

94. $\sqrt{3}\sin\theta+\cos\theta=1$

Divide each side by 2:

$$\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta=\frac{1}{2}$$

Rewrite in the sum of two angles form using $\cos\phi=\frac{\sqrt{3}}{2}$, $\sin\phi=\frac{1}{2}$, and $\phi=\frac{\pi}{6}$:

$$\sin\theta\cos\phi+\cos\theta\sin\phi=\frac{1}{2}$$

$$\sin(\theta+\phi)=\frac{1}{2}$$

$$\theta+\phi=\frac{\pi}{6} \quad\text{or}\quad \theta+\phi=\frac{5\pi}{6}$$

$$\theta+\frac{\pi}{6}=\frac{\pi}{6} \quad\text{or}\quad \theta+\frac{\pi}{6}=\frac{5\pi}{6}$$

$$\theta=0 \quad\text{or}\quad \theta=\frac{2\pi}{3}$$

The solution set is $\left\{0,\ \frac{2\pi}{3}\right\}$.

95. $\sin\theta+\cos\theta=\sqrt{2}$

Divide each side by $\sqrt{2}$:

$$\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=1$$

Rewrite in the sum of two angles form using $\cos\phi=\frac{1}{\sqrt{2}}$, $\sin\phi=\frac{1}{\sqrt{2}}$, and $\phi=\frac{\pi}{4}$:

$$\sin\theta\cos\phi+\cos\theta\sin\phi=1$$

$$\sin(\theta+\phi)=1$$

$$\theta+\phi=\frac{\pi}{2}$$

$$\theta+\frac{\pi}{4}=\frac{\pi}{2}$$

$$\theta=\frac{\pi}{4}$$

The solution set is $\left\{\frac{\pi}{4}\right\}$.

96. $\sin\theta-\cos\theta=-\sqrt{2}$

Divide each side by $\sqrt{2}$:

$$\frac{1}{\sqrt{2}}\sin\theta-\frac{1}{\sqrt{2}}\cos\theta=-1$$

Rewrite in the sum of two angles form using $\cos\phi=\frac{1}{\sqrt{2}}$, $\sin\phi=\frac{1}{\sqrt{2}}$, and $\phi=\frac{\pi}{4}$:

$$\sin\theta\cos\phi-\sin\phi\cos\theta=-1$$

$$\sin(\theta-\phi)=-1$$

$$\theta-\phi=\frac{3\pi}{2}$$

$$\theta-\frac{\pi}{4}=\frac{3\pi}{2}$$

$$\theta=\frac{7\pi}{4}$$

The solution set is $\left\{\frac{7\pi}{4}\right\}$.

97. $\tan\theta+\sqrt{3}=\sec\theta$

$$\frac{\sin\theta}{\cos\theta}+\sqrt{3}=\frac{1}{\cos\theta}$$

$$\sin\theta+\sqrt{3}\cos\theta=1$$

$$\sin\theta+\sqrt{3}\cos\theta=1$$

Divide each side by 2:

$$\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta=\frac{1}{2}$$

Rewrite in the difference of two angles form using $\cos\phi=\frac{1}{2}$, $\sin\phi=\frac{\sqrt{3}}{2}$, and $\phi=\frac{\pi}{3}$:

$$\sin\theta\cos\phi+\cos\theta\sin\phi=\frac{1}{2}$$

$$\sin(\theta+\phi)=\frac{1}{2}$$

$$\theta+\phi=\frac{\pi}{6} \quad\text{or}\quad \theta+\phi=\frac{5\pi}{6}$$

$$\theta+\frac{\pi}{3}=\frac{\pi}{6} \qquad \theta+\frac{\pi}{3}=\frac{5\pi}{6}$$

$$\theta=-\frac{\pi}{6}=\frac{11\pi}{6} \qquad \theta=\frac{\pi}{2}$$

But since $\frac{\pi}{2}$ is not in the domain of the tangent function then the solution set is $\left\{\frac{11\pi}{6}\right\}$.

98. $\cot\theta + \csc\theta = -\sqrt{3}$

$$\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = -\sqrt{3}$$

$$\cos\theta + 1 = -\sqrt{3}\sin\theta$$

$$\sqrt{3}\sin\theta + \cos\theta = -1$$

Divide each side by 2:

$$\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = -\frac{1}{2}$$

Rewrite in the sum of two angles form using $\cos\phi = \frac{\sqrt{3}}{2}$, $\sin\phi = \frac{1}{2}$, and $\phi = \frac{\pi}{6}$:

$$\sin\theta\cos\phi + \cos\theta\sin\phi = -\frac{1}{2}$$

$$\sin(\theta + \phi) = -\frac{1}{2}$$

$$\theta + \phi = \frac{7\pi}{6} \quad \text{or} \quad \theta + \phi = \frac{11\pi}{6}$$

$$\theta + \frac{\pi}{6} = \frac{7\pi}{6} \quad \text{or} \quad \theta + \frac{\pi}{6} = \frac{11\pi}{6}$$

$$\theta = \pi \quad \text{or} \quad \theta = \frac{5\pi}{3}$$

But since π is not in the domain of the cotangent function then the solution set is $\left\{\frac{5\pi}{3}\right\}$.

99. Let $\alpha = \sin^{-1} v$ and $\beta = \cos^{-1} v$. Then $\sin\alpha = v = \cos\beta$, and since $\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$, $\cos\left(\frac{\pi}{2} - \alpha\right) = \cos\beta$. If $v \geq 0$, then $0 \leq \alpha \leq \frac{\pi}{2}$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left[0, \frac{\pi}{2}\right]$. If $v < 0$, then $-\frac{\pi}{2} \leq \alpha < 0$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left(\frac{\pi}{2}, \pi\right]$. Either way, $\cos\left(\frac{\pi}{2} - \alpha\right) = \cos\beta$ implies $\frac{\pi}{2} - \alpha = \beta$, or $\alpha + \beta = \frac{\pi}{2}$. Thus, $\sin^{-1} v + \cos^{-1} v = \frac{\pi}{2}$.

100. Let $\alpha = \tan^{-1} v$ and $\beta = \cot^{-1} v$. Then $\tan\alpha = v = \cot\beta$, and since $\tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right)$, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$. If $v \geq 0$, then $0 \leq \alpha < \frac{\pi}{2}$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left(0, \frac{\pi}{2}\right]$. If $v < 0$, then $-\frac{\pi}{2} < \alpha < 0$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left(\frac{\pi}{2}, \pi\right)$. Either way, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$ implies $\frac{\pi}{2} - \alpha = \beta$, or $\alpha + \beta = \frac{\pi}{2}$. Thus, $\tan^{-1} v + \cot^{-1} v = \frac{\pi}{2}$. Note that $v \neq 0$ since $\cot^{-1} 0$ is undefined.

101. Let $\alpha = \tan^{-1}\left(\frac{1}{v}\right)$ and $\beta = \tan^{-1} v$. Because $\frac{1}{v}$ must be defined, $v \neq 0$ and so $\alpha, \beta \neq 0$. Then $\tan\alpha = \frac{1}{v} = \frac{1}{\tan\beta} = \cot\beta$, and since $\tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right)$, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$.

Because $v > 0$, $0 < \alpha < \frac{\pi}{2}$ and so $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left(0, \frac{\pi}{2}\right)$. Then $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$ implies $\frac{\pi}{2} - \alpha = \beta$ or $\alpha = \frac{\pi}{2} - \beta$. Thus,

$$\tan^{-1}\left(\frac{1}{v}\right) = \frac{\pi}{2} - \tan^{-1} v, \text{ if } v > 0.$$

102. Let $\theta = \tan^{-1} e^{-v}$. Then $\tan\theta = e^{-v}$, so $\cot\theta = \frac{1}{e^{-v}} = e^{v}$. Because $0 < \theta < \frac{\pi}{2}$, we know that $e^{-v} > 0$, which means $\cot^{-1} e^{v} = \cot^{-1}\left(\cot\theta\right) = \theta = \tan^{-1} e^{-v}$.

103. $\sin\left(\sin^{-1} v + \cos^{-1} v\right)$

$= \sin\left(\sin^{-1} v\right)\cos\left(\cos^{-1} v\right) + \cos\left(\sin^{-1} v\right)\sin\left(\cos^{-1} v\right)$

$= v \cdot v + \sqrt{1-v^2}\sqrt{1-v^2}$

$= v^2 + 1 - v^2$

$= 1$

104. $\cos\left(\sin^{-1} v + \cos^{-1} v\right)$

$= \cos\left(\sin^{-1} v\right)\cos\left(\cos^{-1} v\right) - \sin\left(\sin^{-1} v\right)\sin\left(\cos^{-1} v\right)$

$= \sqrt{1-v^2} \cdot v - v \cdot \sqrt{1-v^2}$

$= 0$

105. $\dfrac{f(x+h)-f(x)}{h}$

$= \dfrac{\sin(x+h)-\sin x}{h}$

$= \dfrac{\sin x\cos h + \cos x \sin h - \sin x}{h}$

$= \dfrac{\cos x \sin h - \sin x + \sin x \cos h}{h}$

$= \dfrac{\cos x \sin h - \sin x\left(1-\cos h\right)}{h}$

$= \cos x \cdot \dfrac{\sin h}{h} - \sin x \cdot \dfrac{1-\cos h}{h}$

106. $\dfrac{f(x+h)-f(x)}{h}$

$\dfrac{\cos(x+h)-\cos x}{h}$

$= \dfrac{\cos x \cos h - \sin x \sin h - \cos x}{h}$

$= \dfrac{-\sin x \sin h + \cos x \cos h - \cos x}{h}$

$= \dfrac{-\sin x \sin h - \cos x\left(1-\cos h\right)}{h}$

$= -\sin x \cdot \dfrac{\sin h}{h} - \cos x \cdot \dfrac{1-\cos h}{h}$

107. a. $\tan\left(\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3\right) = \tan\left(\left(\tan^{-1} 1 + \tan^{-1} 2\right) + \tan^{-1} 3\right) = \dfrac{\tan\left(\tan^{-1} 1 + \tan^{-1} 2\right) + \tan\left(\tan^{-1} 3\right)}{1 - \tan\left(\tan^{-1} 1 + \tan^{-1} 2\right)\tan\left(\tan^{-1} 3\right)}$

$$= \frac{\dfrac{\tan\left(\tan^{-1} 1\right) + \tan\left(\tan^{-1} 2\right)}{1-\tan\left(\tan^{-1} 1\right)\tan\left(\tan^{-1} 2\right)} + 3}{1 - \dfrac{\tan\left(\tan^{-1} 1\right) + \tan\left(\tan^{-1} 2\right)}{1-\tan\left(\tan^{-1} 1\right)\tan\left(\tan^{-1} 2\right)} \cdot 3} = \frac{\dfrac{1+2}{1-1\cdot 2} + 3}{1 - \dfrac{1+2}{1-1\cdot 2}\cdot 3} = \frac{\dfrac{3}{-1} + 3}{1 - \dfrac{3}{-1}\cdot 3} = \frac{-3+3}{1+9} = \frac{0}{10} = 0$$

b. From the definition of the inverse tangent function we know $0 < \tan^{-1} 1 < \dfrac{\pi}{2}$, $0 < \tan^{-1} 2 < \dfrac{\pi}{2}$, and $0 < \tan^{-1} 3 < \dfrac{\pi}{2}$. Thus, $0 < \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 < \dfrac{3\pi}{2}$. On the interval $\left(0, \dfrac{3\pi}{2}\right)$, $\tan\theta = 0$ if and only if $\theta = \pi$. Therefore, from part (a), $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.

108. $\cos\phi\sin^2(\omega t)-\sin\phi\sin(\omega t)\cos(\omega t)=\sin(\omega t)\big(\cos\phi\sin(\omega t)-\sin\phi\cos(\omega t)\big)$

$$=\sin(\omega t)\big(\sin(\omega t)\cos\phi-\cos(\omega t)\sin\phi\big)$$

$$=\sin(\omega t)\sin(\omega t-\phi)$$

109. Note that $\theta=\theta_2-\theta_1$.

Then $\tan\theta=\tan(\theta_2-\theta_1)=\dfrac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_2\tan\theta_1}=\dfrac{m_2-m_1}{1+m_2m_1}$

110. $\sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta)$

$$=(\sin\alpha\cos\theta-\cos\alpha\sin\theta)(\sin\beta\cos\theta-\cos\beta\sin\theta)(\sin\gamma\cos\theta-\cos\gamma\sin\theta)$$

$$=\sin\theta\left(\sin\alpha\left(\frac{\cos\theta}{\sin\theta}\right)-\cos\alpha\right)\sin\theta\left(\sin\beta\left(\frac{\cos\theta}{\sin\theta}\right)-\cos\beta\right)\sin\theta\left(\sin\gamma\left(\frac{\cos\theta}{\sin\theta}\right)-\cos\gamma\right)$$

$$=\sin^3\theta\left(\sin\alpha\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\alpha}{\sin\alpha}\right)\right)\left(\sin\beta\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\beta}{\sin\beta}\right)\right)\left(\sin\gamma\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\gamma}{\sin\gamma}\right)\right)$$

$$=\sin^3\theta\big(\sin\alpha(\cot\theta-\cot\alpha)\big)\big(\sin\beta(\cot\theta-\cot\beta)\big)\big(\sin\gamma(\cot\theta-\cot\gamma)\big)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma(\cot\beta+\cot\gamma)(\cot\alpha+\cot\gamma)(\cot\alpha+\cot\beta)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\cos\beta}{\sin\beta}+\frac{\cos\gamma}{\sin\gamma}\right)\left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\gamma}{\sin\gamma}\right)\left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}\right)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\sin(\gamma+\beta)}{\sin\beta\sin\gamma}\right)\left(\frac{\sin(\gamma+\alpha)}{\sin\alpha\sin\gamma}\right)\left(\frac{\sin(\beta+\alpha)}{\sin\alpha\sin\beta}\right)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\sin(180^\circ-\alpha)}{\sin\beta\sin\gamma}\right)\left(\frac{\sin(180^\circ-\beta)}{\sin\alpha\sin\gamma}\right)\left(\frac{\sin(180^\circ-\gamma)}{\sin\alpha\sin\beta}\right)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\sin\alpha}{\sin\beta\sin\gamma}\right)\left(\frac{\sin\beta}{\sin\alpha\sin\gamma}\right)\left(\frac{\sin\gamma}{\sin\alpha\sin\beta}\right)$$

$$=\sin^3\theta$$

111. If $\tan\alpha=x+1$ and $\tan\beta=x-1$, then

$$2\cot(\alpha-\beta)=2\cdot\frac{1}{\tan(\alpha-\beta)}$$

$$=\frac{2}{\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}$$

$$=\frac{2(1+\tan\alpha\tan\beta)}{\tan\alpha-\tan\beta}$$

$$=\frac{2\big(1+(x+1)(x-1)\big)}{x+1-(x-1)}$$

$$=\frac{2\big(1+(x^2-1)\big)}{x+1-x+1}$$

$$=\frac{2x^2}{2}$$

$$=x^2$$

112. The first step in the derivation,

$\tan\left(\theta+\dfrac{\pi}{2}\right)=\dfrac{\tan\theta+\tan\frac{\pi}{2}}{1-\tan\theta\cdot\tan\frac{\pi}{2}}$, is impossible

because $\tan\dfrac{\pi}{2}$ is undefined.

113. If formula (7) is used, we obtain

$\tan\left(\dfrac{\pi}{2}-\theta\right)=\dfrac{\tan\frac{\pi}{2}-\tan\theta}{1+\tan\frac{\pi}{2}\cdot\tan\theta}$. However, this is

impossible because $\tan\dfrac{\pi}{2}$ is undefined. Using formulas (3a) and (3b), we obtain

$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}.$$
$$=\frac{\cos\theta}{\sin\theta}$$
$$=\cot\theta$$

114. $x^2+5x+1=-2x^2-11x-4$

$3x^2+16x+5=0$

$(3x+1)(x+5)=0$

$3x+1=0$ *or* $x+5=0$

$x=-\frac{1}{3}$ $\qquad x=-5$

For $x=-\frac{1}{3}$

$y=\left(-\frac{1}{3}\right)^2+5\left(-\frac{1}{3}\right)+1$

$=\frac{1}{9}-\frac{5}{3}+1=-\frac{5}{9}$

For $x=-5$

$y=(-5)^2+5(-5)+1$

$=25-25+1=1$

The intersection points are:

$\left(-\frac{1}{3},-\frac{5}{9}\right),(-5,1)$

115. $\frac{17\pi}{6}\cdot\frac{180}{\pi}=510°$

116. $45°=\frac{\pi}{4}$ radians

$A=\frac{1}{2}r^2\theta=\frac{1}{2}(6)^2\left(\frac{\pi}{4}\right)$

$=\frac{36\pi}{8}=\frac{9\pi}{2}\approx 14.14 \text{ cm}^2$

117. $\tan\theta=-2$ and $270°<\theta<360°$ (quadrant IV)

Using the Pythagorean Identities:

$\sec^2\theta=\tan^2\theta+1$

$\sec^2\theta=(-2)^2+1=4+1=5$

$\sec\theta=\pm\sqrt{5}$

Note that $\sec\theta$ must be positive since θ lies in quadrant IV. Thus, $\sec\theta=\sqrt{5}$.

$\cos\theta=\frac{1}{\sec\theta}=\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{5}}{5}$

$\tan\theta=\frac{\sin\theta}{\cos\theta}$, so

$\sin\theta=(\tan\theta)(\cos\theta)=-2\left(\frac{\sqrt{5}}{5}\right)=-\frac{2\sqrt{5}}{5}$.

$\csc\theta=\frac{1}{\sin\theta}=\frac{1}{-\frac{2\sqrt{5}}{5}}=-\frac{5}{2\sqrt{5}}=-\frac{\sqrt{5}}{2}$

$\cot\theta=\frac{1}{\tan\theta}=\frac{1}{-2}=-\frac{1}{2}$

Section 7.6

1. $\sin^2\theta$, $2\cos^2\theta$, $2\sin^2\theta$

2. $1-\cos\theta$

3. $\sin\theta$

4. True

5. False, only the first one is equivalent.

6. False, you cannot add the arguments or tan.

7. b

8. c

9. $\sin\theta=\frac{3}{5}$, $0<\theta<\frac{\pi}{2}$. Thus, $0<\frac{\theta}{2}<\frac{\pi}{4}$, which means $\frac{\theta}{2}$ lies in quadrant I.

   $y=3,\ r=5$

   $x^2+3^2=5^2,\ x>0$

   $x^2=25-9=16,\ x>0$

   $x=4$

   So, $\cos\theta=\frac{4}{5}$ and $\tan\theta=\frac{3}{4}$.

   a. $\sin(2\theta)=2\sin\theta\cos\theta=2\cdot\frac{3}{5}\cdot\frac{4}{5}=\frac{24}{25}$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$$= \sqrt{\frac{1-\frac{4}{5}}{2}} = \sqrt{\frac{\frac{1}{5}}{2}} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

d. $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

$$= \sqrt{\frac{1+\frac{4}{5}}{2}} = \sqrt{\frac{\frac{9}{5}}{2}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{1-\frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{24}{7}$$

f. The angle is in QI so

$$\tan\left(\frac{\theta}{2}\right) = +\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\frac{4}{5}}{1+\frac{4}{5}}}$$

$$= \sqrt{\frac{\frac{1}{5}}{\frac{9}{5}}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

10. $\cos\theta = \frac{3}{5},\ 0 < \theta < \frac{\pi}{2}$. Thus, $0 < \frac{\theta}{2} < \frac{\pi}{4}$, which means $\frac{\theta}{2}$ lies in quadrant I.

$x = 3,\ r = 5$

$3^2 + y^2 = 5^2,\ y > 0$

$y^2 = 25 - 9 = 16,\ y > 0$

$y = 4$

So, $\sin\theta = \frac{4}{5}$ and $\tan\theta = \frac{4}{3}$.

a. $\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\frac{4}{5}\cdot\frac{3}{5} = \frac{24}{25}$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$$= \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{\frac{2}{5}}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

d. $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

$$= \sqrt{\frac{1+\frac{3}{5}}{2}} = \sqrt{\frac{\frac{8}{5}}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{1-\frac{16}{9}} = \frac{\frac{8}{3}}{-\frac{7}{9}} = -\frac{24}{7}$$

f. The angle is in QI so

$$\tan\left(\frac{\theta}{2}\right) = +\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\frac{3}{5}}{1+\frac{3}{5}}}$$

$$= \sqrt{\frac{\frac{2}{5}}{\frac{8}{5}}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

11. $\tan\theta = \frac{4}{3},\quad \pi < \theta < \frac{3\pi}{2}$. Thus, $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$, which means $\frac{\theta}{2}$ lies in quadrant II.

$x = -3,\ y = -4$

$r^2 = (-3)^2 + (-4)^2 = 9 + 16 = 25$

$r = 5$

$\sin\theta = -\frac{4}{5},\quad \cos\theta = -\frac{3}{5},\quad \tan\theta = \frac{4}{3}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\cdot\left(-\frac{4}{5}\right)\cdot\left(-\frac{3}{5}\right) = \frac{24}{25}$$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(-\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$

$=\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}$

$=\sqrt{\frac{\frac{8}{5}}{2}}=\sqrt{\frac{4}{5}}=\frac{2}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$

d. $\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}$

$=-\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}}$

$=-\sqrt{\frac{\frac{2}{5}}{2}}=-\sqrt{\frac{1}{5}}=-\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}=-\frac{\sqrt{5}}{5}$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}=\frac{\frac{8}{3}}{1-\frac{16}{9}}=\frac{\frac{8}{3}}{-\frac{7}{9}}=-\frac{24}{7}$

f. The angle is in QII so

$\tan\left(\frac{\theta}{2}\right)=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=-\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{1+\left(-\frac{3}{5}\right)}}$

$=-\sqrt{\frac{\frac{8}{5}}{\frac{2}{5}}}=-\sqrt{4}=-2$

12. $\tan\theta=\frac{1}{2},\quad \pi<\theta<\frac{3\pi}{2}$. Thus, $\frac{\pi}{2}<\frac{\theta}{2}<\frac{3\pi}{4}$,

which means $\frac{\theta}{2}$ lie in quadrant II.

$x=-2,\ y=-1$

$r^2=(-2)^2+(-1)^2=4+1=5$

$r=\sqrt{5}$

$\sin\theta=-\frac{1}{\sqrt{5}}=-\frac{\sqrt{5}}{5},\ \cos\theta=-\frac{2}{\sqrt{5}}=-\frac{2\sqrt{5}}{5}$

$\tan\theta=\frac{1}{2}$

a. $\sin(2\theta)=2\sin\theta\cos\theta$

$=2\cdot\left(-\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{2\sqrt{5}}{5}\right)=\frac{4}{5}$

b. $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$=\left(-\frac{2\sqrt{5}}{5}\right)^2-\left(-\frac{\sqrt{5}}{5}\right)^2$

$=\frac{20}{25}-\frac{5}{25}=\frac{15}{25}=\frac{3}{5}$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1-\left(\frac{-2\sqrt{5}}{5}\right)}{2}}$

$=\sqrt{\frac{\frac{5+2\sqrt{5}}{5}}{2}}$

$=\sqrt{\frac{5+2\sqrt{5}}{10}}$

d. $\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}=-\sqrt{\frac{1+\left(\frac{-2\sqrt{5}}{5}\right)}{2}}$

$=-\sqrt{\frac{\frac{5-2\sqrt{5}}{5}}{2}}$

$=-\sqrt{\frac{5-2\sqrt{5}}{10}}$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}=\frac{1}{1-\frac{1}{4}}=\frac{1}{\frac{3}{4}}=\frac{4}{3}$

f. The angle is in QII so

$$\tan\left(\frac{\theta}{2}\right)=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=-\sqrt{\frac{1-\left(-\frac{2}{\sqrt{5}}\right)}{1+\left(-\frac{2}{\sqrt{5}}\right)}}$$

$$=-\sqrt{\frac{\frac{\sqrt{5}+2}{\sqrt{5}}}{\frac{\sqrt{5}-2}{\sqrt{5}}}}=-\sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}}$$

$$=-\sqrt{\frac{\left(5+2\sqrt{5}\right)}{\left(5-2\sqrt{5}\right)}\cdot\frac{\left(5+2\sqrt{5}\right)}{\left(5+2\sqrt{5}\right)}}$$

$$=-\sqrt{\frac{25+20\sqrt{5}+20}{25-20}}=-\sqrt{\frac{45+20\sqrt{5}}{5}}$$

$$=-\sqrt{9+4\sqrt{5}}$$

13. $\cos\theta=-\frac{\sqrt{6}}{3},\ \ \frac{\pi}{2}<\theta<\pi$. Thus, $\frac{\pi}{4}<\frac{\theta}{2}<\frac{\pi}{2}$, which means $\frac{\theta}{2}$ lies in quadrant I.

$x=-\sqrt{6},\ r=3$

$\left(-\sqrt{6}\right)^2+y^2=3^2$

$y^2=9-6=3$

$y=\sqrt{3}$

$\sin\theta=\frac{\sqrt{3}}{3}$ and $\tan\theta=-\frac{\sqrt{2}}{2}$

a. $\sin(2\theta)=2\sin\theta\cos\theta$

$$=2\cdot\left(\frac{\sqrt{3}}{3}\right)\cdot\left(-\frac{\sqrt{6}}{3}\right)$$

$$=-\frac{2\sqrt{18}}{9}=-\frac{6\sqrt{2}}{9}=-\frac{2\sqrt{2}}{3}$$

b. $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$$=\left(-\frac{\sqrt{6}}{3}\right)^2-\left(\frac{\sqrt{3}}{3}\right)^2$$

$$=\frac{6}{9}-\frac{3}{9}=\frac{3}{9}=\frac{1}{3}$$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1-\left(-\frac{\sqrt{6}}{3}\right)}{2}}$

$$=\sqrt{\frac{\frac{3+\sqrt{6}}{3}}{2}}$$

$$=\sqrt{\frac{3+\sqrt{6}}{6}}$$

d. $\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}=\sqrt{\frac{1+\left(-\frac{\sqrt{6}}{3}\right)}{2}}$

$$=\sqrt{\frac{\frac{3-\sqrt{6}}{3}}{2}}$$

$$=\sqrt{\frac{3-\sqrt{6}}{6}}$$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$$=\frac{2\left(-\frac{\sqrt{2}}{2}\right)}{1-\left(\frac{\sqrt{2}}{2}\right)^2}=\frac{-\sqrt{2}}{1-\frac{1}{2}}=-\frac{\sqrt{2}}{\frac{1}{2}}=-2\sqrt{2}$$

f. The angle is in QI so

$$\tan\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\sqrt{\frac{1-\left(-\frac{\sqrt{6}}{3}\right)}{1+\left(-\frac{\sqrt{6}}{3}\right)}}$$

$$=\sqrt{\frac{\frac{3+\sqrt{6}}{3}}{\frac{3-\sqrt{6}}{3}}}=\sqrt{\frac{3+\sqrt{6}}{3-\sqrt{6}}}$$

$$=\sqrt{\frac{\left(3+\sqrt{6}\right)}{\left(3-\sqrt{6}\right)}\cdot\frac{\left(3+\sqrt{6}\right)}{\left(3+\sqrt{6}\right)}}=\sqrt{\frac{9+6\sqrt{6}+6}{9-6}}$$

$$=\sqrt{\frac{15+6\sqrt{6}}{3}}=\sqrt{5+2\sqrt{6}}$$

14. $\sin\theta = -\frac{\sqrt{3}}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$. Thus, $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$, which means $\frac{\theta}{2}$ lies in quadrant II.

$y = -\sqrt{3},\ r = 3$

$x^2 + \left(-\sqrt{3}\right)^2 = 3$

$x^2 = 9 - 3 = 6$

$x = \sqrt{6}$

$\cos\theta = \frac{\sqrt{6}}{3}$ and $\tan\theta = -\frac{\sqrt{2}}{2}$

a. $$\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\left(-\frac{\sqrt{3}}{3}\right)\cdot\left(\frac{\sqrt{6}}{3}\right) = -\frac{2\sqrt{18}}{9} = -\frac{6\sqrt{2}}{9} = -\frac{2\sqrt{2}}{3}$$

b. $$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{\sqrt{6}}{3}\right)^2 - \left(-\frac{\sqrt{3}}{3}\right)^2 = \frac{6}{9} - \frac{3}{9} = \frac{3}{9} = \frac{1}{3}$$

c. $$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{\sqrt{6}}{3}}{2}} = \sqrt{\frac{\frac{3-\sqrt{6}}{3}}{2}} = \sqrt{\frac{3-\sqrt{6}}{6}}$$

d. $$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\frac{\sqrt{6}}{3}}{2}} = -\sqrt{\frac{\frac{3+\sqrt{6}}{3}}{2}} = -\sqrt{\frac{3+\sqrt{6}}{6}}$$

e. $$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{\sqrt{2}}{2}\right)}{1-\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{-\sqrt{2}}{1-\frac{1}{2}} = -\frac{\sqrt{2}}{\frac{1}{2}} = -2\sqrt{2}$$

f. The angle is in QII so

$$\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-\left(\frac{\sqrt{6}}{3}\right)}{1+\left(\frac{\sqrt{6}}{3}\right)}}$$
$$= -\sqrt{\frac{\frac{3-\sqrt{6}}{3}}{\frac{3+\sqrt{6}}{3}}} = -\sqrt{\frac{3-\sqrt{6}}{3+\sqrt{6}}}$$
$$= -\sqrt{\frac{\left(3-\sqrt{6}\right)}{\left(3+\sqrt{6}\right)}\cdot\frac{\left(3-\sqrt{6}\right)}{\left(3-\sqrt{6}\right)}}$$
$$= -\sqrt{\frac{9-6\sqrt{6}+6}{9-6}} = -\sqrt{\frac{15-6\sqrt{6}}{3}}$$
$$= -\sqrt{5-2\sqrt{6}}$$

15. $\sec\theta = 3$, $\sin\theta > 0$, so $0 < \theta < \frac{\pi}{2}$. Thus, $0 < \frac{\theta}{2} < \frac{\pi}{4}$, which means $\frac{\theta}{2}$ lies in quadrant I.

$\cos\theta = \frac{1}{3}$, $x = 1$, $r = 3$.

$1^2 + y^2 = 3^2$

$y^2 = 9 - 1 = 8$

$y = \sqrt{8} = 2\sqrt{2}$

$\sin\theta = \frac{2\sqrt{2}}{3}$ and $\tan\theta = 2\sqrt{2}$

a. $$\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\frac{2\sqrt{2}}{3}\cdot\frac{1}{3} = \frac{4\sqrt{2}}{9}$$

b. $$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{1}{3}\right)^2 - \left(\frac{2\sqrt{2}}{3}\right)^2 = \frac{1}{9} - \frac{8}{9} = -\frac{7}{9}$$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$

$=\sqrt{\frac{1-\frac{1}{3}}{2}}=\sqrt{\frac{\frac{2}{3}}{2}}=\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{3}$

d. $\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$

$=\sqrt{\frac{1+\frac{1}{3}}{2}}=\sqrt{\frac{\frac{4}{3}}{2}}=\sqrt{\frac{2}{3}}=\frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{6}}{3}$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{2\left(2\sqrt{2}\right)}{1-\left(2\sqrt{2}\right)^2}=\frac{4\sqrt{2}}{1-8}=-\frac{4\sqrt{2}}{7}$

f. The angle is in QI so

$\tan\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\sqrt{\frac{1-\left(\frac{1}{3}\right)}{1+\left(\frac{1}{3}\right)}}$

$=\sqrt{\frac{\frac{2}{3}}{\frac{4}{3}}}=\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$

16. $\csc\theta=-\sqrt{5},\ \cos\theta<0$, so $\pi<\theta<\frac{3\pi}{2}$. Thus, $\frac{\pi}{2}<\frac{\theta}{2}<\frac{3\pi}{4}$, which means $\frac{\theta}{2}$ lies in quadrant II.

$\sin\theta=\frac{-1}{\sqrt{5}}=-\frac{\sqrt{5}}{5}$, $r=\sqrt{5},\ y=-1$

$x^2+(-1)^2=\left(\sqrt{5}\right)^2$

$x^2=5-1=4$

$x=-2$

$\cos\theta=\frac{-2}{\sqrt{5}}=-\frac{2\sqrt{5}}{5}$ and $\tan\theta=\frac{1}{2}$

a. $\sin(2\theta)=2\sin\theta\cos\theta$

$=2\cdot\left(-\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{2\sqrt{5}}{5}\right)=\frac{4}{5}$

b. $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$=\left(-\frac{2\sqrt{5}}{5}\right)^2-\left(-\frac{\sqrt{5}}{5}\right)^2$

$=\frac{20}{25}-\frac{5}{25}=\frac{15}{25}=\frac{3}{5}$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1-\left(-\frac{2\sqrt{5}}{5}\right)}{2}}$

$=\sqrt{\frac{\frac{5+2\sqrt{5}}{5}}{2}}$

$=\sqrt{\frac{5+2\sqrt{5}}{10}}$

d. $\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}=-\sqrt{\frac{1+\left(-\frac{2\sqrt{5}}{5}\right)}{2}}$

$=-\sqrt{\frac{\frac{5-2\sqrt{5}}{5}}{2}}$

$=-\sqrt{\frac{5-2\sqrt{5}}{10}}$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}=\frac{1}{1-\frac{1}{4}}=\frac{1}{\frac{3}{4}}=\frac{4}{3}$

f. The angle is in QII so

$\tan\left(\frac{\theta}{2}\right)=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=-\sqrt{\frac{1-\left(-\frac{2}{\sqrt{5}}\right)}{1+\left(-\frac{2}{\sqrt{5}}\right)}}$

$=-\sqrt{\frac{\frac{\sqrt{5}+2}{\sqrt{5}}}{\frac{\sqrt{5}-2}{\sqrt{5}}}}=-\sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}}=-\sqrt{9+4\sqrt{5}}$

17. $\cot\theta = -2,\ \sec\theta < 0$, so $\frac{\pi}{2} < \theta < \pi$. Thus,

$\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$, which means $\frac{\theta}{2}$ lies in quadrant I.

$x = -2,\ y = 1$

$r^2 = (-2)^2 + 1^2 = 4 + 1 = 5$

$r = \sqrt{5}$

$\sin\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$,

$\cos\theta = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$, $\tan\theta = -\frac{1}{2}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\cdot\left(\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{2\sqrt{5}}{5}\right) = -\frac{20}{25} = -\frac{4}{5}$$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(-\frac{2\sqrt{5}}{5}\right)^2 - \left(\frac{\sqrt{5}}{5}\right)^2$$

$$= \frac{20}{25} - \frac{5}{25} = \frac{15}{25} = \frac{3}{5}$$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{2\sqrt{5}}{5}\right)}{2}}$

$$= \sqrt{\frac{\frac{5+2\sqrt{5}}{5}}{2}}$$

$$= \sqrt{\frac{5+2\sqrt{5}}{10}}$$

d. $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\left(-\frac{2\sqrt{5}}{5}\right)}{2}}$

$$= \sqrt{\frac{\frac{5-2\sqrt{5}}{5}}{2}}$$

$$= \sqrt{\frac{5-2\sqrt{5}}{10}}$$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)^2} = \frac{-1}{1-\frac{1}{4}} = -\frac{1}{\frac{3}{4}} = -\frac{4}{3}$$

f. The angle is in QI so

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\left(-\frac{2}{\sqrt{5}}\right)}{1+\left(-\frac{2}{\sqrt{5}}\right)}}$$

$$= \sqrt{\frac{\frac{\sqrt{5}+2}{\sqrt{5}}}{\frac{\sqrt{5}-2}{\sqrt{5}}}} = \sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}}$$

$$= \sqrt{\frac{\left(5+2\sqrt{5}\right)}{\left(5-2\sqrt{5}\right)}\cdot\frac{\left(5+2\sqrt{5}\right)}{\left(5+2\sqrt{5}\right)}}$$

$$= \sqrt{\frac{25+40\sqrt{5}+20}{25-20}} = \sqrt{\frac{45+40\sqrt{5}}{5}}$$

$$= \sqrt{9+4\sqrt{5}}$$

18. $\sec\theta = 2,\ \csc\theta < 0$, so $\frac{3\pi}{2} < \theta < 2\pi$. Thus,

$\frac{3\pi}{4} < \frac{\theta}{2} < \pi$, which means $\frac{\theta}{2}$ lies in quadrant II.

$\cos\theta = \frac{1}{2},\ x = 1,\ r = 2$

$1^2 + y^2 = 2^2$

$y^2 = 4 - 1 = 3$

$y = \sqrt{3}$

$\sin\theta = -\frac{\sqrt{3}}{2}$ and $\tan\theta = -\sqrt{3}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\cdot\left(-\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}$$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{1}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$

$=\sqrt{\frac{1-\frac{1}{2}}{2}}=\sqrt{\frac{\frac{1}{2}}{2}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$

d. $\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}$

$=-\sqrt{\frac{1+\frac{1}{2}}{2}}=-\sqrt{\frac{\frac{3}{2}}{2}}=-\sqrt{\frac{3}{4}}=-\frac{\sqrt{3}}{2}$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{2(-\sqrt{3})}{1-(-\sqrt{3})^2}=\frac{-2\sqrt{3}}{1-3}=\sqrt{3}$

f. The angle is in QII so

$\tan\left(\frac{\theta}{2}\right)=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=-\sqrt{\frac{1-\left(\frac{1}{2}\right)}{1+\left(\frac{1}{2}\right)}}$

$=-\sqrt{\frac{\frac{1}{2}}{\frac{3}{2}}}=-\sqrt{\frac{1}{3}}=-\frac{\sqrt{3}}{3}$

19. $\tan\theta=-3,\ \sin\theta<0$, so $\frac{3\pi}{2}<\theta<2\pi$. Thus,

$\frac{3\pi}{4}<\frac{\theta}{2}<\pi$, which means $\frac{\theta}{2}$ lies in quadrant II.

$x=1,\quad y=-3$

$r^2=1^2+(-3)^2=1+9=10$

$r=\sqrt{10}$

$\sin\theta=\frac{-3}{\sqrt{10}}=-\frac{3\sqrt{10}}{10},\quad \cos\theta=\frac{1}{\sqrt{10}}=\frac{\sqrt{10}}{10},$

$\tan\theta=-3$

a. $\sin(2\theta)=2\sin\theta\cos\theta$

$=2\cdot\left(-\frac{3\sqrt{10}}{10}\right)\cdot\left(\frac{\sqrt{10}}{10}\right)$

$=-\frac{6}{10}=-\frac{3}{5}$

b. $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$=\left(\frac{\sqrt{10}}{10}\right)^2-\left(-\frac{3\sqrt{10}}{10}\right)^2$

$=\frac{10}{100}-\frac{90}{100}=-\frac{80}{100}=-\frac{4}{5}$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1-\frac{\sqrt{10}}{10}}{2}}$

$=\sqrt{\frac{\frac{10-\sqrt{10}}{10}}{2}}$

$=\sqrt{\frac{10-\sqrt{10}}{20}}$

$=\frac{1}{2}\sqrt{\frac{10-\sqrt{10}}{5}}$

d. $\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}=-\sqrt{\frac{1+\frac{\sqrt{10}}{10}}{2}}$

$=-\sqrt{\frac{\frac{10+\sqrt{10}}{10}}{2}}$

$=-\sqrt{\frac{10+\sqrt{10}}{20}}$

$=-\frac{1}{2}\sqrt{\frac{10+\sqrt{10}}{5}}$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{2(-3)}{1-(-3)^2}=\frac{-6}{1-9}=-\frac{6}{-8}=\frac{3}{4}$

f. The angle is in QII so

$$\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-\left(\frac{1}{\sqrt{10}}\right)}{1+\left(\frac{1}{\sqrt{10}}\right)}}$$

$$= -\sqrt{\frac{\sqrt{10}-1}{\sqrt{10}+1}} = -\sqrt{\frac{10-\sqrt{10}}{10+\sqrt{10}}}$$

$$= -\sqrt{\frac{\left(10-\sqrt{10}\right)}{\left(10+\sqrt{10}\right)} \cdot \frac{\left(10-\sqrt{10}\right)}{\left(10-\sqrt{10}\right)}}$$

$$= -\sqrt{\frac{100-20\sqrt{10}+10}{100-10}} = -\sqrt{\frac{110-20\sqrt{10}}{90}}$$

$$= -\frac{\sqrt{11-2\sqrt{10}}}{3}$$

20. $\cot\theta = 3,\ \cos\theta < 0$, so $\pi < \theta < \frac{3\pi}{2}$. Thus,

$\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$ which means $\frac{\theta}{2}$ is in quadrant II.

$x = -3,\ \ y = -1$

$r^2 = (-3)^2 + (-1)^2 = 9 + 1 = 10$

$r = \sqrt{10}$

$\sin\theta = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$,

$\cos\theta = -\frac{3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$ and $\tan\theta = \frac{1}{3}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\cdot\left(-\frac{\sqrt{10}}{10}\right)\cdot\left(-\frac{3\sqrt{10}}{10}\right) = \frac{6}{10} = \frac{3}{5}$$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(-\frac{3\sqrt{10}}{10}\right)^2 - \left(-\frac{\sqrt{10}}{10}\right)^2$$

$$= \frac{90}{100} - \frac{10}{100} = \frac{80}{100} = \frac{4}{5}$$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{3\sqrt{10}}{10}\right)}{2}}$

$$= \sqrt{\frac{\frac{10+3\sqrt{10}}{10}}{2}}$$

$$= \sqrt{\frac{10+3\sqrt{10}}{20}}$$

$$= \frac{1}{2}\sqrt{\frac{10+3\sqrt{10}}{5}}$$

d. $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{3\sqrt{10}}{10}\right)}{2}}$

$$= -\sqrt{\frac{\frac{10-3\sqrt{10}}{10}}{2}}$$

$$= -\sqrt{\frac{10-3\sqrt{10}}{20}}$$

$$= -\frac{1}{2}\sqrt{\frac{10-3\sqrt{10}}{5}}$$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{3}{4}$$

f. The angle is in QII so

$$\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-\left(-\frac{3}{\sqrt{10}}\right)}{1+\left(-\frac{3}{\sqrt{10}}\right)}}$$

$$= -\sqrt{\frac{\sqrt{10}+3}{\sqrt{10}-3}} = -\sqrt{\frac{10+3\sqrt{10}}{10-3\sqrt{10}}}$$

$$= -\sqrt{\frac{\left(10+3\sqrt{10}\right)}{\left(10-3\sqrt{10}\right)} \cdot \frac{\left(10+3\sqrt{10}\right)}{\left(10+3\sqrt{10}\right)}}$$

$$= -\sqrt{\frac{100+60\sqrt{10}+90}{100-90}} = -\sqrt{\frac{190+60\sqrt{10}}{10}}$$

$$= -\sqrt{19+6\sqrt{10}}$$

21. $\sin 22.5° = \sin\left(\frac{45°}{2}\right)$

$= \sqrt{\frac{1-\cos 45°}{2}}$

$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

22. $\cos 22.5° = \cos\left(\frac{45°}{2}\right)$

$= \sqrt{\frac{1+\cos 45°}{2}}$

$= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

23. $\tan\frac{7\pi}{8} = \tan\left(\frac{\frac{7\pi}{4}}{2}\right) = -\sqrt{\frac{1-\cos\frac{7\pi}{4}}{1+\cos\frac{7\pi}{4}}}$

$= -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\cdot\frac{2}{2}}$

$= -\sqrt{\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\cdot\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)}$

$= -\sqrt{\frac{\left(2-\sqrt{2}\right)^2}{2}}$

$= -\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)$

$= -\left(\sqrt{2}-1\right)$

$= 1-\sqrt{2}$

24. $\tan\frac{9\pi}{8} = \tan\left(\frac{\frac{9\pi}{4}}{2}\right) = \sqrt{\frac{1-\cos\frac{9\pi}{4}}{1+\cos\frac{9\pi}{4}}}$

$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\cdot\frac{2}{2}}$

$= \sqrt{\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\cdot\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)}$

$= \sqrt{\frac{\left(2-\sqrt{2}\right)^2}{2}}$

$= \frac{2-\sqrt{2}}{\sqrt{2}}$

$= \sqrt{2}-1$

$= -1+\sqrt{2}$

25. $\cos 165° = \cos\left(\frac{330°}{2}\right)$

$= -\sqrt{\frac{1+\cos 330°}{2}}$

$= -\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2+\sqrt{3}}{4}} = -\frac{\sqrt{2+\sqrt{3}}}{2}$

26. $\sin 195° = \sin\left(\frac{390°}{2}\right) = -\sqrt{\frac{1-\cos 390°}{2}}$

$= -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$

$= -\sqrt{\frac{2-\sqrt{3}}{4}}$

$= -\frac{\sqrt{2-\sqrt{3}}}{2}$

27. $\sec\frac{15\pi}{8}=\frac{1}{\cos\frac{15\pi}{8}}=\frac{1}{\cos\left(\frac{\frac{15\pi}{4}}{2}\right)}$

$=\frac{1}{\sqrt{\frac{1+\cos\frac{15\pi}{4}}{2}}}$

$=\frac{1}{\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}}$

$=\frac{1}{\sqrt{\frac{2+\sqrt{2}}{4}}}$

$=\frac{2}{\sqrt{2+\sqrt{2}}}$

$=\left(\frac{2}{\sqrt{2+\sqrt{2}}}\right)\cdot\left(\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)$

$=\left(\frac{2\sqrt{2+\sqrt{2}}}{2+\sqrt{2}}\right)\cdot\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)$

$=\frac{2\left(2-\sqrt{2}\right)\sqrt{2+\sqrt{2}}}{2}$

$=\left(2-\sqrt{2}\right)\sqrt{2+\sqrt{2}}$

28. $\csc\frac{7\pi}{8}=\frac{1}{\sin\frac{7\pi}{8}}=\frac{1}{\sin\left(\frac{\frac{7\pi}{4}}{2}\right)}$

$=\frac{1}{\sqrt{\frac{1-\cos\frac{7\pi}{4}}{2}}}$

$=\frac{1}{\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}}$

$=\frac{1}{\sqrt{\frac{2-\sqrt{2}}{4}}}$

$=\frac{2}{\sqrt{2-\sqrt{2}}}$

$=\left(\frac{2}{\sqrt{2-\sqrt{2}}}\right)\cdot\left(\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\right)$

$=\left(\frac{2\sqrt{2-\sqrt{2}}}{2-\sqrt{2}}\right)\cdot\left(\frac{2+\sqrt{2}}{2+\sqrt{2}}\right)$

$=\frac{2\left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}}{2}$

$=\left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}$

29. $\sin\left(-\frac{\pi}{8}\right)=\sin\left(\frac{-\frac{\pi}{4}}{2}\right)$

$=-\sqrt{\frac{1-\cos\left(-\frac{\pi}{4}\right)}{2}}$

$=-\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}=-\sqrt{\frac{2-\sqrt{2}}{4}}=-\frac{\sqrt{2-\sqrt{2}}}{2}$

30. $\cos\left(-\frac{3\pi}{8}\right)=\cos\left(\frac{-\frac{3\pi}{4}}{2}\right)$

$=\sqrt{\frac{1+\cos\left(-\frac{3\pi}{4}\right)}{2}}$

$=\sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}}=\sqrt{\frac{2-\sqrt{2}}{4}}=\frac{\sqrt{2-\sqrt{2}}}{2}$

31. θ lies in quadrant II. Since $x^2+y^2=5$, $r=\sqrt{5}$. Now, the point $(a, 2)$ is on the circle, so

$$a^2+2^2=5$$
$$a^2=5-2^2$$
$$a=-\sqrt{5-2^2}=-\sqrt{1}=-1$$

(a is negative because θ lies in quadrant II.)

Thus, $\sin\theta=\frac{b}{r}=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$ and

$\cos\theta=\frac{a}{r}=\frac{-1}{\sqrt{5}}=-\frac{\sqrt{5}}{5}$. Thus,

$$f(2\theta)=\sin(2\theta)=2\sin\theta\cos\theta$$
$$=2\cdot\left(\frac{2\sqrt{5}}{5}\right)\cdot\left(-\frac{\sqrt{5}}{5}\right)=-\frac{20}{25}=-\frac{4}{5}$$

32. From the solution to Problem 29, we have $\sin\theta=\frac{2\sqrt{5}}{5}$ and $\cos\theta=-\frac{\sqrt{5}}{5}$.

Thus, $g(2\theta)=\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$$=\left(-\frac{\sqrt{5}}{5}\right)^2-\left(\frac{2\sqrt{5}}{5}\right)^2$$
$$=\frac{5}{25}-\frac{20}{25}=-\frac{15}{25}=-\frac{3}{5}$$

33. Note: Since θ lies in quadrant II, $\frac{\theta}{2}$ must lie in quadrant I. Therefore, $\cos\frac{\theta}{2}$ is positive. From the solution to Problem 29, we have $\cos\theta=-\frac{\sqrt{5}}{5}$.

Thus, $g\left(\frac{\theta}{2}\right)=\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$

$$=\sqrt{\frac{1+\left(\frac{-\sqrt{5}}{5}\right)}{2}}$$
$$=\sqrt{\frac{\frac{5-\sqrt{5}}{5}}{2}}$$
$$=\sqrt{\frac{5-\sqrt{5}}{10}}$$
$$=\frac{\sqrt{10\left(5-\sqrt{5}\right)}}{10}$$

34. Note: Since θ lies in quadrant II, $\frac{\theta}{2}$ must lie in quadrant I. Therefore, $\sin\frac{\theta}{2}$ is positive. From the solution to Problem 29, we have $\cos\theta=-\frac{\sqrt{5}}{5}$.

Thus, $f\left(\frac{\theta}{2}\right)=\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$

$$=\sqrt{\frac{1-\left(\frac{-\sqrt{5}}{5}\right)}{2}}$$
$$=\sqrt{\frac{\frac{5+\sqrt{5}}{5}}{2}}$$
$$=\sqrt{\frac{5+\sqrt{5}}{10}}$$
$$=\frac{\sqrt{10\left(5+\sqrt{5}\right)}}{10}$$

35. θ lies in quadrant II. Since $x^2+y^2=5$, $r=\sqrt{5}$. Now, the point $(a, 2)$ is on the circle, so

$$a^2+2^2=5$$
$$a^2=5-2^2$$
$$a=-\sqrt{5-2^2}=-\sqrt{1}=-1$$

(a is negative because θ lies in quadrant II.)

Thus, $\tan\theta=\frac{b}{a}=\frac{2}{-1}=-2$.

$$h(2\theta)=\tan(2\theta)$$
$$=\frac{2\tan\theta}{1-\tan^2\theta}$$
$$=\frac{2(-2)}{1-(-2)^2}=\frac{-4}{1-4}=\frac{-4}{-3}=\frac{4}{3}$$

36. From the solution to Problem 29, we have $\sin\theta = \frac{2\sqrt{5}}{5}$ and $\cos\theta = -\frac{\sqrt{5}}{5}$. Thus,

$$h\left(\frac{\theta}{2}\right) = \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\left(\frac{-\sqrt{5}}{5}\right)}{\frac{2\sqrt{5}}{5}}$$
$$= \frac{\frac{5+\sqrt{5}}{5}}{\frac{2\sqrt{5}}{5}}$$
$$= \frac{5+\sqrt{5}}{2\sqrt{5}}$$
$$= \frac{5+\sqrt{5}}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{5\sqrt{5}+5}{10}$$
$$= \frac{\sqrt{5}+1}{2} = \frac{1+\sqrt{5}}{2}$$

37. α lies in quadrant III. Since $x^2 + y^2 = 1$, $r = \sqrt{1} = 1$. Now, the point $\left(-\frac{1}{4}, b\right)$ is on the circle, so

$$\left(-\frac{1}{4}\right)^2 + b^2 = 1$$
$$b^2 = 1-\left(-\frac{1}{4}\right)^2$$
$$b = -\sqrt{1-\left(-\frac{1}{4}\right)^2} = -\sqrt{\frac{15}{16}} = -\frac{\sqrt{15}}{4}$$

(b is negative because α lies in quadrant III.)

Thus, $\cos\alpha = \frac{a}{r} = \frac{-\frac{1}{4}}{1} = -\frac{1}{4}$ and

$\sin\alpha = \frac{b}{r} = \frac{-\frac{\sqrt{15}}{4}}{1} = -\frac{\sqrt{15}}{4}$. Thus,

$$g(2\alpha) = \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$
$$= \left(-\frac{1}{4}\right)^2 - \left(\frac{\sqrt{15}}{4}\right)^2$$
$$= \frac{1}{16} - \frac{15}{16} = -\frac{14}{16} = -\frac{7}{8}$$

38. From the solution to Problem 37, we have $\sin\alpha = -\frac{\sqrt{15}}{4}$ and $\cos\alpha = -\frac{1}{4}$. Thus,

$$f(2\alpha) = \sin(2\alpha)$$
$$= 2\sin\alpha\cos\alpha$$
$$= 2\cdot\left(-\frac{\sqrt{15}}{4}\right)\cdot\left(-\frac{1}{4}\right) = \frac{\sqrt{15}}{8}$$

39. Note: Since α lies in quadrant III, $\frac{\alpha}{2}$ must lie in quadrant II. Therefore, $\sin\frac{\alpha}{2}$ is positive. From the solution to Problem 37, we have $\cos\alpha = -\frac{1}{4}$.

Thus, $f\left(\frac{\alpha}{2}\right) = \sin\frac{\alpha}{2}$

$$= \sqrt{\frac{1-\cos\alpha}{2}}$$
$$= \sqrt{\frac{1-\left(-\frac{1}{4}\right)}{2}}$$
$$= \sqrt{\frac{\frac{5}{4}}{2}} = \sqrt{\frac{5}{8}} = \sqrt{\frac{5}{8}\cdot\frac{2}{2}} = \sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{4}$$

40. Note: Since α lies in quadrant III, $\frac{\alpha}{2}$ must lie in quadrant II. Therefore, $\cos\frac{\alpha}{2}$ is negative. From the solution to Problem 37, we have $\cos\alpha = -\frac{1}{4}$.

Thus,

$$g\left(\frac{\alpha}{2}\right) = \cos\frac{\alpha}{2}$$
$$= -\sqrt{\frac{1+\cos\alpha}{2}}$$
$$= -\sqrt{\frac{1+\left(-\frac{1}{4}\right)}{2}}$$
$$= -\sqrt{\frac{\frac{3}{4}}{2}} = -\sqrt{\frac{3}{8}} = -\sqrt{\frac{3}{8}\cdot\frac{2}{2}} = -\sqrt{\frac{6}{16}} = -\frac{\sqrt{6}}{4}$$

41. From the solution to Problem 37, we have $\sin\alpha = -\frac{\sqrt{15}}{4}$ and $\cos\alpha = -\frac{1}{4}$. Thus,

$$h\left(\frac{\alpha}{2}\right) = \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{1}{4}\right)}{-\frac{\sqrt{15}}{4}} = \frac{\frac{5}{4}}{-\frac{\sqrt{15}}{4}} = -\frac{5}{\sqrt{15}} = -\frac{5}{\sqrt{15}}\cdot\frac{\sqrt{15}}{\sqrt{15}} = -\frac{5\sqrt{15}}{15} = -\frac{\sqrt{15}}{3}$$

42. α lies in quadrant III. Since $x^2 + y^2 = 1$, $r = \sqrt{1} = 1$. Now, the point $\left(-\frac{1}{4}, b\right)$ is on the circle, so

$$\left(-\frac{1}{4}\right)^2 + b^2 = 1$$

$$b^2 = 1 - \left(-\frac{1}{4}\right)^2$$

$$b = -\sqrt{1-\left(-\frac{1}{4}\right)^2} = -\sqrt{\frac{15}{16}} = -\frac{\sqrt{15}}{4}$$

(b is negative because α lies in quadrant III.)

Thus, $\tan\theta = \frac{b}{a} = \frac{-\frac{\sqrt{15}}{4}}{-\frac{1}{4}} = \sqrt{15}$.

$$h(2\alpha) = \tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(\sqrt{15}\right)}{1-\left(\sqrt{15}\right)^2} = \frac{2\sqrt{15}}{1-15} = \frac{2\sqrt{15}}{-14} = -\frac{\sqrt{15}}{7}$$

43. $\sin^4\theta = \left(\sin^2\theta\right)^2$

$$= \left(\frac{1-\cos(2\theta)}{2}\right)^2$$
$$= \frac{1}{4}\left[1 - 2\cos(2\theta) + \cos^2(2\theta)\right]$$
$$= \frac{1}{4} - \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta)$$
$$= \frac{1}{4} - \frac{1}{2}\cos(2\theta) + \frac{1}{4}\left(\frac{1+\cos(4\theta)}{2}\right)$$
$$= \frac{1}{4} - \frac{1}{2}\cos(2\theta) + \frac{1}{8} + \frac{1}{8}\cos(4\theta)$$
$$= \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$$

44. $\sin(4\theta) = \sin(2\cdot 2\theta)$

$$= 2\sin(2\theta)\cos(2\theta)$$
$$= 2(2\sin\theta\cos\theta)\left(1 - 2\sin^2\theta\right)$$
$$= 4\sin\theta\cos\theta\left(1 - 2\sin^2\theta\right)$$
$$= (\cos\theta)\left[4\sin\theta\left(1 - 2\sin^2\theta\right)\right]$$
$$= (\cos\theta)\left(4\sin\theta - 8\sin^3\theta\right)$$

45. $\cos(3\theta) = \cos(2\theta + \theta)$

$$= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta$$
$$= \left(2\cos^2\theta - 1\right)\cos\theta - 2\sin\theta\cos\theta\sin\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\left(1 - \cos^2\theta\right)\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta$$

46. $\cos(4\theta) = \cos(2\cdot 2\theta)$

$$= 2\cos^2(2\theta) - 1$$
$$= 2\left(2\cos^2\theta - 1\right)^2 - 1$$
$$= 2\left(4\cos^4\theta - 4\cos^2\theta + 1\right) - 1$$
$$= 8\cos^4\theta - 8\cos^2\theta + 2 - 1$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

47. We use the result of problem 44 to help solve this problem:

$$\begin{aligned}
\sin(5\theta) &= \sin(4\theta+\theta)\\
&= \sin(4\theta)\cos\theta + \cos(4\theta)\sin\theta\\
&= \cos\theta\left(4\sin\theta - 8\sin^3\theta\right)\cos\theta + \cos(2(2\theta))\sin\theta\\
&= \cos^2\theta\left(4\sin\theta - 8\sin^3\theta\right) + \left(1-2\sin^2(2\theta)\right)\sin\theta\\
&= \left(1-\sin^2\theta\right)\left(4\sin\theta - 8\sin^3\theta\right)\\
&\qquad + \sin\theta\left(1-2\left(2\sin\theta\cos\theta\right)^2\right)\\
&= 4\sin\theta - 12\sin^3\theta + 8\sin^5\theta\\
&\qquad + \sin\theta\left(1-8\sin^2\theta\cos^2\theta\right)\\
&= 4\sin\theta - 12\sin^3\theta + 8\sin^5\theta\\
&\qquad + \sin\theta - 8\sin^3\theta\left(1-\sin^2\theta\right)\\
&= 5\sin\theta - 12\sin^3\theta + 8\sin^5\theta - 8\sin^3\theta + 8\sin^5\theta\\
&= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta
\end{aligned}$$

48. We use the results from problems 44 and 46 to help solve this problem:

$$\begin{aligned}
\cos(5\theta) &= \cos(4\theta+\theta)\\
&= \cos(4\theta)\cos\theta - \sin(4\theta)\sin\theta\\
&= \left(8\cos^4\theta - 8\cos^2\theta + 1\right)\cos\theta\\
&\qquad - \left(\cos\theta\left(4\sin\theta - 8\sin^3\theta\right)\right)\sin\theta\\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta\\
&\qquad - 4\cos\theta\sin^2\theta + 8\cos\theta\sin^4\theta\\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta\\
&\qquad - 4\cos\theta(1-\cos^2\theta) + 8\cos\theta(1-\cos^2\theta)^2\\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta - 4\cos\theta\\
&\qquad + 4\cos^3\theta + 8\cos\theta(1-2\cos^2\theta+\cos^4\theta)\\
&= 8\cos^5\theta - 4\cos^3\theta - 3\cos\theta\\
&\qquad + 8\cos\theta - 16\cos^3\theta + 8\cos^5\theta\\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}$$

49.
$$\begin{aligned}
\cos^4\theta - \sin^4\theta &= \left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right)\\
&= 1\cdot\cos(2\theta)\\
&= \cos(2\theta)
\end{aligned}$$

50.
$$\begin{aligned}
\frac{\cot\theta-\tan\theta}{\cot\theta+\tan\theta} &= \frac{\dfrac{\cos\theta}{\sin\theta}-\dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\cos\theta}}\\
&= \frac{\dfrac{\cos^2\theta-\sin^2\theta}{\sin\theta\cos\theta}}{\dfrac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}}\\
&= \frac{\cos^2\theta-\sin^2\theta}{\sin\theta\cos\theta}\cdot\frac{\sin\theta\cos\theta}{\cos^2\theta+\sin^2\theta}\\
&= \frac{\cos^2\theta-\sin^2\theta}{1}\\
&= \cos(2\theta)
\end{aligned}$$

51.
$$\begin{aligned}
\cot(2\theta) = \frac{1}{\tan(2\theta)} &= \frac{1}{\dfrac{2\tan\theta}{1-\tan^2\theta}}\\
&= \frac{1-\tan^2\theta}{2\tan\theta}\\
&= \frac{1-\dfrac{1}{\cot^2\theta}}{\dfrac{2}{\cot\theta}}\\
&= \frac{\dfrac{\cot^2\theta-1}{\cot^2\theta}}{\dfrac{2}{\cot\theta}}\\
&= \frac{\cot^2\theta-1}{\cot^2\theta}\cdot\frac{\cot\theta}{2}\\
&= \frac{\cot^2\theta-1}{2\cot\theta}
\end{aligned}$$

52.
$$\begin{aligned}
\cot(2\theta) = \frac{1}{\tan(2\theta)} &= \frac{1}{\dfrac{2\tan\theta}{1-\tan^2\theta}}\\
&= \frac{1-\tan^2\theta}{2\tan\theta}\\
&= \frac{1}{2}\left(\frac{1}{\tan\theta}-\frac{\tan^2\theta}{\tan\theta}\right)\\
&= \frac{1}{2}\left(\cot\theta-\tan\theta\right)
\end{aligned}$$

53. $\sec(2\theta)=\dfrac{1}{\cos(2\theta)}=\dfrac{1}{2\cos^2\theta-1}$

$$=\dfrac{1}{\dfrac{2}{\sec^2\theta}-1}$$

$$=\dfrac{1}{\dfrac{2-\sec^2\theta}{\sec^2\theta}}$$

$$=\dfrac{\sec^2\theta}{2-\sec^2\theta}$$

54. $\csc(2\theta)=\dfrac{1}{\sin(2\theta)}=\dfrac{1}{2\sin\theta\cos\theta}$

$$=\dfrac{1}{2}\cdot\dfrac{1}{\cos\theta}\cdot\dfrac{1}{\sin\theta}$$

$$=\dfrac{1}{2}\sec\theta\csc\theta$$

55. $\cos^2(2u)-\sin^2(2u)=\cos\left[2(2u)\right]=\cos(4u)$

56. $(4\sin u\cos u)(1-2\sin^2 u)$

$=2(2\sin u\cos u)(1-2\sin^2 u)$

$=2\sin 2u\cos 2u$

$=\sin(2\cdot 2u)$

$=\sin(4u)$

57. $\dfrac{\cos(2\theta)}{1+\sin(2\theta)}=\dfrac{\cos^2\theta-\sin^2\theta}{1+2\sin\theta\cos\theta}$

$$=\dfrac{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta}$$

$$=\dfrac{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{(\cos\theta+\sin\theta)(\cos\theta+\sin\theta)}$$

$$=\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$$

$$=\dfrac{\dfrac{\cos\theta-\sin\theta}{\sin\theta}}{\dfrac{\cos\theta+\sin\theta}{\sin\theta}}$$

$$=\dfrac{\dfrac{\cos\theta}{\sin\theta}-\dfrac{\sin\theta}{\sin\theta}}{\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\sin\theta}}$$

$$=\dfrac{\cot\theta-1}{\cot\theta+1}$$

58. $\sin^2\theta\cos^2\theta=\dfrac{1}{4}\left(4\sin^2\theta\cos^2\theta\right)$

$$=\dfrac{1}{4}\left(2\sin\theta\cos\theta\right)^2$$

$$=\dfrac{1}{4}\left[\sin(2\theta)\right]^2$$

$$=\dfrac{1}{4}\cdot\left[\dfrac{1-\cos(4\theta)}{2}\right]$$

$$=\dfrac{1}{8}\left[1-\cos(4\theta)\right]$$

59. $\sec^2\left(\dfrac{\theta}{2}\right)=\dfrac{1}{\cos^2\left(\dfrac{\theta}{2}\right)}=\dfrac{1}{\dfrac{1+\cos\theta}{2}}=\dfrac{2}{1+\cos\theta}$

60. $\csc^2\left(\dfrac{\theta}{2}\right)=\dfrac{1}{\sin^2\left(\dfrac{\theta}{2}\right)}=\dfrac{1}{\dfrac{1-\cos\theta}{2}}=\dfrac{2}{1-\cos\theta}$

61. $\cot^2\left(\dfrac{v}{2}\right)=\dfrac{1}{\tan^2\left(\dfrac{v}{2}\right)}=\dfrac{1}{\dfrac{1-\cos v}{1+\cos v}}$

$$=\dfrac{1+\cos v}{1-\cos v}$$

$$=\dfrac{1+\dfrac{1}{\sec v}}{1-\dfrac{1}{\sec v}}$$

$$=\dfrac{\dfrac{\sec v+1}{\sec v}}{\dfrac{\sec v-1}{\sec v}}$$

$$=\dfrac{\sec v+1}{\sec v}\cdot\dfrac{\sec v}{\sec v-1}$$

$$=\dfrac{\sec v+1}{\sec v-1}$$

62. $\tan\frac{v}{2}=\frac{1-\cos v}{\sin v}=\frac{1}{\sin v}-\frac{\cos v}{\sin v}=\csc v-\cot v$

63. $$\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{1-\frac{1-\cos\theta}{1+\cos\theta}}{1+\frac{1-\cos\theta}{1+\cos\theta}}$$
$$=\frac{\frac{1+\cos\theta-(1-\cos\theta)}{1+\cos\theta}}{\frac{1+\cos\theta+1-\cos\theta}{1+\cos\theta}}$$
$$=\frac{\frac{2\cos\theta}{1+\cos\theta}}{\frac{2}{1+\cos\theta}}$$
$$=\frac{2\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{2}$$
$$=\cos\theta$$

64. $$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}$$
$$=\frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta}$$
$$=\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta$$
$$=(\sin^2\theta+\cos^2\theta)-\frac{1}{2}(2\sin\theta\cos\theta)$$
$$=1-\frac{1}{2}\sin(2\theta)$$

65. $$\frac{\sin(3\theta)}{\sin\theta}-\frac{\cos(3\theta)}{\cos\theta}=\frac{\sin(3\theta)\cos\theta-\cos(3\theta)\sin\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta}$$
$$=\frac{\sin 2\theta}{\sin\theta\cos\theta}$$
$$=\frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$
$$=2$$

66. $$\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}-\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\frac{(\cos\theta+\sin\theta)^2-(\cos\theta-\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$
$$=\frac{\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta-(\cos^2\theta-2\cos\theta\sin\theta+\sin^2\theta)}{\cos^2\theta-\sin^2\theta}$$
$$=\frac{\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta-\cos^2\theta+2\cos\theta\sin\theta-\sin^2\theta}{\cos^2\theta-\sin^2\theta}$$
$$=\frac{4\cos\theta\sin\theta}{\cos(2\theta)}$$
$$=\frac{2(2\sin\theta\cos\theta)}{\cos(2\theta)}$$
$$=\frac{2\sin(2\theta)}{\cos(2\theta)}$$
$$=2\tan(2\theta)$$

67. $\tan(3\theta)=\tan(2\theta+\theta)$

$$=\frac{\tan(2\theta)+\tan\theta}{1-\tan(2\theta)\tan\theta}=\frac{\frac{2\tan\theta}{1-\tan^2\theta}+\tan\theta}{1-\frac{2\tan\theta}{1-\tan^2\theta}\cdot\tan\theta}=\frac{\frac{2\tan\theta+\tan\theta-\tan^3\theta}{1-\tan^2\theta}}{\frac{1-\tan^2\theta-2\tan^2\theta}{1-\tan^2\theta}}=\frac{3\tan\theta-\tan^3\theta}{1-\tan^2\theta}\cdot\frac{1-\tan^2\theta}{1-3\tan^2\theta}=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

68. $\tan\theta+\tan(\theta+120^\circ)+\tan(\theta+240^\circ)$

$$=\tan\theta+\frac{\tan\theta+\tan 120^\circ}{1-\tan\theta\tan 120^\circ}+\frac{\tan\theta+\tan 240^\circ}{1-\tan\theta\tan 240^\circ}$$

$$=\tan\theta+\frac{\tan\theta-\sqrt{3}}{1-\tan\theta\left(-\sqrt{3}\right)}+\frac{\tan\theta+\sqrt{3}}{1-\tan\theta\left(\sqrt{3}\right)}$$

$$=\tan\theta+\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}+\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}$$

$$=\frac{\tan\theta\left(1-3\tan^2\theta\right)+\left(\tan\theta-\sqrt{3}\right)\left(1-\sqrt{3}\tan\theta\right)+\left(\tan\theta+\sqrt{3}\right)\left(1+\sqrt{3}\tan\theta\right)}{1-3\tan^2\theta}$$

$$=\frac{\tan\theta-3\tan^3\theta+\tan\theta-\sqrt{3}\tan^2\theta-\sqrt{3}+3\tan\theta+\tan\theta+\sqrt{3}\tan^2\theta+\sqrt{3}+3\tan\theta}{1-3\tan^2\theta}$$

$$=\frac{-3\tan^3\theta+9\tan\theta}{1-3\tan^2\theta}$$

$$=\frac{3\left(3\tan\theta-\tan^3\theta\right)}{1-3\tan^2\theta}$$

$$=3\tan\left(3\theta\right)\quad\text{(from Problem 65)}$$

69. $\frac{1}{2}\cdot\left(\ln\left|1-\cos\left(2\theta\right)\right|-\ln 2\right)$

$$=\frac{1}{2}\cdot\ln\left|\frac{1-\cos 2\theta}{2}\right|$$

$$=\ln\left(\left|\frac{1-\cos\left(2\theta\right)}{2}\right|^{1/2}\right)$$

$$=\ln\left(\left|\sin^2\theta\right|^{1/2}\right)$$

$$=\ln\left|\sin\theta\right|$$

70. $\frac{1}{2}\cdot\left(\ln\left|1+\cos\left(2\theta\right)\right|-\ln 2\right)$

$$=\frac{1}{2}\cdot\ln\left|\frac{1+\cos 2\theta}{2}\right|$$

$$=\ln\left(\left|\frac{1+\cos\left(2\theta\right)}{2}\right|^{1/2}\right)$$

$$=\ln\left(\left|\cos^2\theta\right|^{1/2}\right)$$

$$=\ln\left|\cos\theta\right|$$

71.

$$\cos\left(2\theta\right)+6\sin^2\theta=4$$
$$1-2\sin^2\theta+6\sin^2\theta=4$$
$$4\sin^2\theta=3$$
$$\sin^2\theta=\frac{3}{4}$$
$$\sin\theta=\pm\frac{\sqrt{3}}{2}$$
$$\theta=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$$

The solution set is $\left\{\frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}\right\}$.

72.

$$\cos\left(2\theta\right)=2-2\sin^2\theta$$
$$1-2\sin^2\theta=2-2\sin^2\theta$$
$$1=2\quad\text{(not possible)}$$

The equation has no real solution.

73.
$$\cos(2\theta) = \cos\theta$$
$$2\cos^2\theta - 1 = \cos\theta$$
$$2\cos^2\theta - \cos\theta - 1 = 0$$
$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$
$2\cos\theta + 1 = 0$ or $\cos\theta - 1 = 0$

$\cos\theta = -\frac{1}{2}$ $\quad$ $\cos\theta = 1$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ $\quad$ $\theta = 0$

The solution set is $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

74.
$$\sin(2\theta) = \cos\theta$$
$$2\sin\theta\cos\theta = \cos\theta$$
$$2\sin\theta\cos\theta - \cos\theta = 0$$
$$(\cos\theta)(2\sin\theta - 1) = 0$$
$\cos\theta = 0$ or $2\sin\theta = 1$

$\cos\theta = 0$ $\quad$ $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

The solution set is $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$.

75.
$$\sin(2\theta) + \sin(4\theta) = 0$$
$$\sin(2\theta) + 2\sin(2\theta)\cos(2\theta) = 0$$
$$\sin(2\theta)\left(1 + 2\cos(2\theta)\right) = 0$$
$\sin(2\theta) = 0$ or $1 + 2\cos(2\theta) = 0$

$\cos(2\theta) = -\frac{1}{2}$

$2\theta = 0 + 2k\pi$ or $2\theta = \pi + 2k\pi$ or

$\theta = k\pi$ $\quad$ $\theta = \frac{\pi}{2} + k\pi$

$2\theta = \frac{2\pi}{3} + 2k\pi$ or $2\theta = \frac{4\pi}{3} + 2k\pi$

$\theta = \frac{\pi}{3} + k\pi$ $\quad$ $\theta = \frac{2\pi}{3} + k\pi$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}\right\}$.

76.
$$\cos(2\theta) + \cos(4\theta) = 0$$
$$\left(2\cos^2\theta - 1\right) + \left(2\cos^2(2\theta) - 1\right) = 0$$
$$2\cos^2\theta - 1 + 2\left[\cos(2\theta)\cos(2\theta)\right] - 1 = 0$$
$$2\cos^2\theta + 2\left(2\cos^2(\theta) - 1\right)\left(2\cos^2(\theta) - 1\right) - 2 = 0$$
$$\left(2\cos^2\theta - 1\right) + 2\left[4\cos^4\theta - 4\cos^2\theta + 1\right] - 1 = 0$$
$$2\cos^2\theta - 1 + 8\cos^4\theta - 8\cos^2\theta + 2 - 1 = 0$$
$$8\cos^4\theta - 6\cos^2\theta = 0$$
$$4\cos^4\theta - 3\cos^2\theta = 0$$
$$\cos^2\theta\left(4\cos^2\theta - 3\right) = 0$$
$\cos^2(\theta) = 0$ or $4\cos^2\theta - 3 = 0$

$\cos\theta = 0$ or $\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$.

77.
$$3 - \sin\theta = \cos(2\theta)$$
$$3 - \sin\theta = 1 - 2\sin^2\theta$$
$$2\sin^2\theta - \sin\theta + 2 = 0$$
This equation is quadratic in $\sin\theta$.
The discriminant is $b^2 - 4ac = 1 - 16 = -15 < 0$.
The equation has no real solutions.

78.
$$\cos(2\theta) + 5\cos\theta + 3 = 0$$
$$2\cos^2\theta - 1 + 5\cos\theta + 3 = 0$$
$$2\cos^2\theta + 5\cos\theta + 2 = 0$$
$$(2\cos\theta + 1)(\cos\theta + 2) = 0$$
$2\cos\theta = -1$ or $\cos\theta = -2$

$\cos\theta = -\frac{1}{2}$ $\quad$ (not possible)

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

The solution set is $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

79.
$$\tan(2\theta) + 2\sin\theta = 0$$
$$\frac{\sin(2\theta)}{\cos(2\theta)} + 2\sin\theta = 0$$
$$\frac{\sin 2\theta + 2\sin\theta\cos 2\theta}{\cos 2\theta} = 0$$
$$2\sin\theta\cos\theta + 2\sin\theta(2\cos^2\theta - 1) = 0$$
$$2\sin\theta\left(\cos\theta + 2\cos^2\theta - 1\right) = 0$$
$$2\sin\theta\left(2\cos^2\theta + \cos\theta - 1\right) = 0$$
$$2\sin\theta(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$2\cos\theta - 1 = 0$ or $2\sin\theta = 0$ or

$\cos\theta = \frac{1}{2}$ $\quad$ $\sin\theta = 0$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ $\quad$ $\theta = 0, \pi$

$\cos\theta + 1 = 0$

$\cos\theta = -1$

$\theta = \pi$

The solution set is $\left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$.

80.
$$\tan(2\theta) + 2\cos\theta = 0$$
$$\frac{\sin(2\theta)}{\cos(2\theta)} + 2\cos\theta = 0$$
$$\frac{\sin(2\theta) + 2\cos\theta\cos 2\theta}{\cos(2\theta)} = 0$$
$$2\sin\theta\cos\theta + 2\cos\theta(1 - 2\sin^2\theta) = 0$$
$$2\cos\theta\left(\sin\theta + 1 - 2\sin^2\theta\right) = 0$$
$$-2\cos\theta\left(2\sin^2\theta - \sin\theta - 1\right) = 0$$
$$-2\cos\theta(2\sin\theta + 1)(\sin\theta - 1) = 0$$

$-2\cos\theta = 0$ or $2\sin\theta + 1 = 0$ or

$\cos\theta = 0$ $\quad$ $\sin\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\sin\theta - 1 = 0$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$

The solution set is $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$.

81. $\sin\left(2\sin^{-1}\frac{1}{2}\right) = \sin\left(2\cdot\frac{\pi}{6}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

82. $\sin\left[2\sin^{-1}\frac{\sqrt{3}}{2}\right] = \sin\left(2\cdot\frac{\pi}{3}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

83.
$$\cos\left(2\sin^{-1}\frac{3}{5}\right) = 1 - 2\sin^2\left(\sin^{-1}\frac{3}{5}\right) = 1 - 2\left(\frac{3}{5}\right)^2 = 1 - \frac{18}{25} = \frac{7}{25}$$

84.
$$\cos\left(2\cos^{-1}\frac{4}{5}\right) = 2\cos^2\left(\cos^{-1}\frac{4}{5}\right) - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = \frac{32}{25} - 1 = \frac{7}{25}$$

85. $\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha = \cos^{-1}\left(-\frac{3}{5}\right)$. α lies in quadrant II.

Then $\cos\alpha = -\frac{3}{5}$, $\frac{\pi}{2} \le \alpha \le \pi$.

$\sec\alpha = -\frac{5}{3}$

$$\tan\alpha = -\sqrt{\sec^2\alpha - 1} = -\sqrt{\left(-\frac{5}{3}\right)^2 - 1} = -\sqrt{\frac{25}{9} - 1} = -\sqrt{\frac{16}{9}} = -\frac{4}{3}$$

$$\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right] = \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$
$$= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}$$
$$= \frac{-\frac{8}{3}}{1-\frac{16}{9}}\cdot\frac{9}{9}$$
$$= \frac{-24}{9-16}$$
$$= \frac{-24}{-7}$$
$$= \frac{24}{7}$$

86. $\tan\left(2\tan^{-1}\frac{3}{4}\right) = \dfrac{2\tan\left(\tan^{-1}\frac{3}{4}\right)}{1-\tan^2\left(\tan^{-1}\frac{3}{4}\right)}$
$$= \frac{2\cdot\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}$$
$$= \frac{\frac{3}{2}}{1-\frac{9}{16}}\cdot\frac{16}{16}$$
$$= \frac{24}{16-9}$$
$$= \frac{24}{7}$$

87. $\sin\left(2\cos^{-1}\frac{4}{5}\right)$

Let $\alpha = \cos^{-1}\frac{4}{5}$. α is in quadrant I.

Then $\cos\alpha = \frac{4}{5},\ 0 \le \alpha \le \frac{\pi}{2}$.

$$\sin\alpha = \sqrt{1-\cos^2\alpha}$$
$$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$
$$\sin\left(2\cos^{-1}\frac{4}{5}\right) = \sin 2\alpha$$
$$= 2\sin\alpha\cos\alpha = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$

88. $\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right]$

Let $\alpha = \tan^{-1}\left(-\frac{4}{3}\right)$. α is in quadrant IV.

Then $\tan\alpha = -\frac{4}{3},\ -\frac{\pi}{2} < \alpha < 0$.

$$\sec\alpha = \sqrt{\tan^2\alpha+1}$$
$$= \sqrt{\left(-\frac{4}{3}\right)^2+1} = \sqrt{\frac{16}{9}+1} = \sqrt{\frac{25}{9}} = \frac{5}{3}$$
$$\cos\alpha = \frac{3}{5}$$
$$\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right] = \cos 2\alpha = 2\cos^2\alpha - 1$$
$$= 2\left(\frac{3}{5}\right)^2 - 1$$
$$= \frac{18}{25} - 1$$
$$= -\frac{7}{25}$$

89. $\sin^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right) = \dfrac{1-\cos\left(\cos^{-1}\frac{3}{5}\right)}{2} = \dfrac{1-\frac{3}{5}}{2}$
$$= \frac{\frac{2}{5}}{2}$$
$$= \frac{1}{5}$$

90. $\cos^2\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$

Let $\alpha = \sin^{-1}\frac{3}{5}$. α is in quadrant I. Then

$\sin\alpha = \frac{3}{5},\ 0 < \alpha < \frac{\pi}{2}$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos^2\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right) = \cos^2\left(\frac{1}{2}\cdot\alpha\right) = \frac{1+\cos\alpha}{2} = \frac{1+\frac{4}{5}}{2} = \frac{\frac{9}{5}}{2} = \frac{9}{10}$$

91. $\sec\left(2\tan^{-1}\frac{3}{4}\right)$

Let $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$. α is in quadrant I.

Then $\tan\alpha = \frac{3}{4}$, $0 < \alpha < \frac{\pi}{2}$.

$$\sec\alpha = \sqrt{\tan^2\alpha+1} = \sqrt{\left(\frac{3}{4}\right)^2+1} = \sqrt{\frac{9}{16}+1} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

$$\cos\alpha = \frac{4}{5}$$

$$\sec\left(2\tan^{-1}\frac{3}{4}\right) = \sec(2\alpha) = \frac{1}{\cos(2\alpha)} = \frac{1}{2\cos^2\alpha-1} = \frac{1}{2\left(\frac{4}{5}\right)^2-1} = \frac{1}{\frac{32}{25}-1} = \frac{1}{\frac{7}{25}} = \frac{25}{7}$$

92. $\csc\left[2\sin^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha = \sin^{-1}\left(-\frac{3}{5}\right)$. α is in quadrant IV.

Then $\sin\alpha = -\frac{3}{5}$, $-\frac{\pi}{2} \le \alpha \le 0$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(-\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\csc\left[2\sin^{-1}\left(-\frac{3}{5}\right)\right] = \csc(2\alpha) = \frac{1}{\sin(2\alpha)} = \frac{1}{2\sin\alpha\cos\alpha} = \frac{1}{2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)} = \frac{1}{-\frac{24}{25}} = -\frac{25}{24}$$

93.

$$f(x) = 0$$
$$\sin(2x) - \sin x = 0$$
$$2\sin x\cos x - \sin x = 0$$
$$\sin x(2\cos x - 1) = 0$$

$\sin x = 0$ or $2\cos x - 1 = 0$

$x = 0, \pi$ $\qquad \cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

The zeros on $0 \le x < 2\pi$ are $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

94.

$$f(x) = 0$$
$$\cos(2x) + \cos x = 0$$
$$2\cos^2 x - 1 + \cos x = 0$$
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$

$2\cos x - 1 = 0$ or $\cos x + 1 = 0$

$\cos x = \frac{1}{2}$ $\qquad \cos x = -1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\qquad x = \pi$

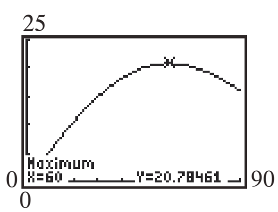
The zeros on $0 \le x < 2\pi$ are $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

95.
$$f(x)=0$$
$$\cos(2x)+\sin^2 x=0$$
$$\cos^2 x-\sin^2 x+\sin^2 x=0$$
$$\cos^2 x=0$$
$$\cos x=0$$
$$x=\frac{\pi}{2},\frac{3\pi}{2}$$
The zeros on $0\le x<2\pi$ are $\frac{\pi}{2},\frac{3\pi}{2}$.

96. a. $\cos(2\theta)+\cos\theta=0$, $0^\circ<\theta<90^\circ$
$$2\cos^2\theta-1+\cos\theta=0$$
$$2\cos^2\theta+\cos\theta-1=0$$
$$(2\cos\theta-1)(\cos\theta+1)=0$$
$$2\cos\theta-1=0 \quad \text{or} \quad \cos\theta+1=0$$
$$\cos\theta=\frac{1}{2} \qquad \cos\theta=-1$$
$$\theta=180^\circ$$
$$\theta=60^\circ, 300^\circ$$
On the interval $0^\circ<\theta<90^\circ$, the solution is 60°.

b. $A(60^\circ)=16\sin(60^\circ)\left[\cos(60^\circ)+1\right]$
$$=16\cdot\frac{\sqrt{3}}{2}\left(\frac{1}{2}+1\right)$$
$$=12\sqrt{3}\text{ in}^2\approx 20.78\text{ in}^2$$

c. Graph $Y_1=16\sin x(\cos x+1)$ and use the MAXIMUM feature:



The maximum area is approximately 20.78 in.^2 when the angle is 60°.

97. a. $D=\dfrac{\frac{1}{2}W}{\csc\theta-\cot\theta}$
$$W=2D(\csc\theta-\cot\theta)$$
$$\csc\theta-\cot\theta=\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}=\frac{1-\cos\theta}{\sin\theta}$$
$$=\tan\frac{\theta}{2}$$
Therefore, $W=2D\tan\frac{\theta}{2}$.

b. Here we have $D=15$ and $W=6.5$.
$$6.5=2(15)\tan\frac{\theta}{2}$$
$$\tan\frac{\theta}{2}=\frac{13}{60}$$
$$\frac{\theta}{2}=\tan^{-1}\frac{13}{60}$$
$$\theta=2\tan^{-1}\frac{13}{60}\approx 24.45^\circ$$

98. $I_x\sin\theta\cos\theta-I_y\sin\theta\cos\theta+I_{xy}\left(\cos^2\theta-\sin^2\theta\right)$
$$=\left(I_x-I_y\right)(\sin\theta\cos\theta)+I_{xy}\left(\cos^2\theta-\sin^2\theta\right)$$
$$=\left(I_x-I_y\right)\frac{1}{2}\sin 2\theta+I_{xy}\cos 2\theta$$
$$=\frac{I_x-I_y}{2}\sin 2\theta+I_{xy}\cos 2\theta$$

99. a. $R(\theta)=\dfrac{v_0^2\sqrt{2}}{16}\cos\theta(\sin\theta-\cos\theta)$
$$=\frac{v_0^2\sqrt{2}}{16}(\cos\theta\sin\theta-\cos^2\theta)$$
$$=\frac{v_0^2\sqrt{2}}{16}\cdot\frac{1}{2}(2\cos\theta\sin\theta-2\cos^2\theta)$$
$$=\frac{v_0^2\sqrt{2}}{32}\left[\sin 2\theta-2\left(\frac{1+\cos 2\theta}{2}\right)\right]$$
$$=\frac{v_0^2\sqrt{2}}{32}\left[\sin(2\theta)-1-\cos(2\theta)\right]$$
$$=\frac{v_0^2\sqrt{2}}{32}\left[\sin(2\theta)-\cos(2\theta)-1\right]$$

b. $\sin(2\theta)+\cos(2\theta)=0$
Divide each side by $\sqrt{2}$:
$$\frac{1}{\sqrt{2}}\sin(2\theta)+\frac{1}{\sqrt{2}}\cos(2\theta)=0$$
Rewrite in the sum of two angles form using

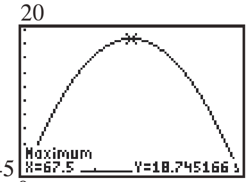
$\cos\phi = \frac{1}{\sqrt{2}}$ and $\sin\phi = \frac{1}{\sqrt{2}}$ and $\phi = \frac{\pi}{4}$:

$$\begin{aligned}\sin(2\theta)\cos\phi + \cos(2\theta)\sin\phi &= 0\\ \sin(2\theta+\phi) &= 0\\ 2\theta+\phi &= 0+k\pi\\ 2\theta+\frac{\pi}{4} &= 0+k\pi\\ 2\theta &= -\frac{\pi}{4}+k\pi\\ \theta &= -\frac{\pi}{8}+\frac{k\pi}{2}\\ \theta &= \frac{3\pi}{8} = 67.5^{\circ}\end{aligned}$$

c. $R = \frac{32^2\sqrt{2}}{32}\left(\sin(2\cdot 67.5^{\circ}) - \cos(2\cdot 67.5^{\circ}) - 1\right)$

$$\begin{aligned}&= 32\sqrt{2}\left(\sin(135^{\circ}) - \cos(135^{\circ}) - 1\right)\\ &= 32\sqrt{2}\left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) - 1\right)\\ &= 32\sqrt{2}\left(\sqrt{2}-1\right)\\ &= 32\left(2-\sqrt{2}\right) \text{ feet} \approx 18.75 \text{ feet}\end{aligned}$$

d. Graph $Y_1 = \frac{32^2\sqrt{2}}{32}\left(\sin(2x) - \cos(2x) - 1\right)$ and use the MAXIMUM feature:



The angle that maximizes the distance is 67.5˚, and the maximum distance is 18.75 feet.

100. $y = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(4\pi x)$

$$\begin{aligned}&= \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(2\cdot 2\pi x)\\ &= \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\left[2\sin(2\pi x)\cos(2\pi x)\right]\\ &= \frac{1}{2}\sin(2\pi x) + \frac{1}{2}\left[\sin(2\pi x)\cos(2\pi x)\right]\\ &= \frac{1}{2}\sin(2\pi x) + \frac{1}{2}\left[\sin(2\pi x)\cdot\left(2\cos^2(\pi x)-1\right)\right]\\ &= \frac{1}{2}\sin(2\pi x) + \sin(2\pi x)\cos^2(\pi x) - \frac{1}{2}\sin(2\pi x)\\ &= \sin(2\pi x)\cos^2(\pi x)\end{aligned}$$

101. Let b represent the base of the triangle.

$\cos\frac{\theta}{2} = \frac{h}{s}$ $\qquad$ $\sin\frac{\theta}{2} = \frac{b/2}{s}$

$h = s\cos\frac{\theta}{2}$ $\qquad$ $b = 2s\sin\frac{\theta}{2}$

$$\begin{aligned}A &= \frac{1}{2}b\cdot h\\ &= \frac{1}{2}\cdot\left(2s\sin\frac{\theta}{2}\right)\left(s\cos\frac{\theta}{2}\right)\\ &= s^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ &= \frac{1}{2}s^2\sin\theta\end{aligned}$$

102. $\sin\theta = \frac{y}{1} = y;\ \cos\theta = \frac{x}{1} = x$

a. $A = 2xy = 2\cos\theta\sin\theta = 2\sin\theta\cos\theta$

b. $2\sin\theta\cos\theta = \sin(2\theta)$

c. The largest value of the sine function is 1. Solve:

$$\begin{aligned}\sin 2\theta &= 1\\ 2\theta &= \frac{\pi}{2}\\ \theta &= \frac{\pi}{4} = 45^{\circ}\end{aligned}$$

d. $x = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\qquad$ $y = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

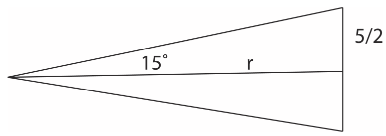
The dimensions of the largest rectangle are $\sqrt{2}$ by $\frac{\sqrt{2}}{2}$.

103. a. $A = 3\cot\left(\frac{\pi}{12}\right)a^2 = 3\cot\left(\frac{\pi}{12}\right)5^2$

$= 3\left(\sqrt{3}+2\right)25 = 75\left(\sqrt{3}+2\right)$

$= 150 + 25\sqrt{3}\ \text{cm}^2$

b. We will use one of the small triangles to compute radius (see figure).



$\tan\left(15°\right) = \frac{5/2}{r}$

$r\tan\left(15°\right) = \frac{5}{2}$

$r\left(2-\sqrt{3}\right) = \frac{5}{2}$

$2r\left(2-\sqrt{3}\right) = 5$

$r = \frac{5}{4-2\sqrt{3}} = \frac{10+5\sqrt{3}}{2}\ \text{cm}$

c. $A = \pi r^2 = \pi\left(\frac{10+5\sqrt{3}}{2}\right)^2$

$= \pi\left(\frac{10+5\sqrt{3}}{2}\right)\left(\frac{10+5\sqrt{3}}{2}\right)$

$= \pi\left(\frac{100+50\sqrt{3}+50\sqrt{3}+75}{4}\right)$

$= \frac{175\pi+100\pi\sqrt{3}}{4}\ \text{cm}^2$

d. $150+75\sqrt{3} - \frac{175\pi+100\pi\sqrt{3}}{4} =$

$\frac{600+300\sqrt{3}-175\pi-100\pi\sqrt{3}}{4}\ \text{cm}^2$

104. a. $\sin\left(2\theta\right) = 2\sin\theta\cos\theta = \frac{2\sin\theta}{\cos\theta}\cdot\frac{\cos^2\theta}{1}$

$= \frac{2\cdot\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}$

$= \frac{2\tan\theta}{\sec^2\theta}$

$= \frac{2\tan\theta}{1+\tan^2\theta}\cdot\frac{4}{4}$

$= \frac{4(2\tan\theta)}{4+(2\tan\theta)^2}$

$= \frac{4x}{4+x^2}$

b. $\cos\left(2\theta\right) = \cos^2\theta - \sin^2\theta = \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$

$= \frac{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}$

$= \frac{1-\tan^2\theta}{1+\tan^2\theta}\cdot\frac{4}{4}$

$= \frac{4-4\tan^2\theta}{4+4\tan^2\theta}$

$= \frac{4-\left(2\tan\theta\right)^2}{4+\left(2\tan\theta\right)^2}$

$= \frac{4-x^2}{4+x^2}$

105. $\frac{1}{2}\cdot\sin^2 x + C = -\frac{1}{4}\cdot\cos\left(2x\right)$

$C = -\frac{1}{4}\cdot\cos\left(2x\right) - \frac{1}{2}\cdot\sin^2 x$

$= -\frac{1}{4}\cdot\left(\cos\left(2x\right)+2\sin^2 x\right)$

$= -\frac{1}{4}\cdot\left(1-2\sin^2 x+2\sin^2 x\right)$

$= -\frac{1}{4}\cdot(1)$

$= -\frac{1}{4}$

106. $\frac{1}{2}\cdot\cos^2 x + C = \frac{1}{4}\cdot\cos(2x)$

$$C = \frac{1}{4}\cdot\cos(2x) - \frac{1}{2}\cdot\cos^2 x$$
$$= \frac{1}{4}\cdot(2\cos^2 x - 1) - \frac{1}{2}\cos^2 x$$
$$= \frac{1}{2}\cos^2 x - \frac{1}{4} - \frac{1}{2}\cos^2 x$$
$$= -\frac{1}{4}$$

107. If $z = \tan\left(\frac{\alpha}{2}\right)$, then

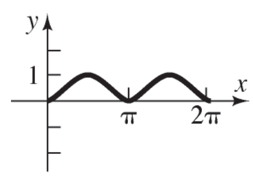
$$\frac{2z}{1+z^2} = \frac{2\tan\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}$$
$$= \frac{2\tan\left(\frac{\alpha}{2}\right)}{\sec^2\left(\frac{\alpha}{2}\right)}$$
$$= 2\tan\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)$$
$$= \frac{2\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}\cdot\cos^2\left(\frac{\alpha}{2}\right)$$
$$= 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)$$
$$= \sin\left[2\left(\frac{\alpha}{2}\right)\right]$$
$$= \sin\alpha$$

108. If $z = \tan\left(\frac{\alpha}{2}\right)$, then

$$\frac{1-z^2}{1+z^2} = \frac{1-\tan^2\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}$$
$$= \frac{1-\frac{1-\cos\alpha}{1+\cos\alpha}}{1+\frac{1-\cos\alpha}{1+\cos\alpha}}$$
$$= \frac{\frac{1+\cos\alpha-(1-\cos\alpha)}{1+\cos\alpha}}{\frac{1+\cos\alpha+1-\cos\alpha}{1+\cos\alpha}}$$
$$= \frac{1+\cos\alpha-(1-\cos\alpha)}{1+\cos\alpha+1-\cos\alpha}$$
$$= \frac{2\cos\alpha}{2}$$
$$= \cos\alpha$$

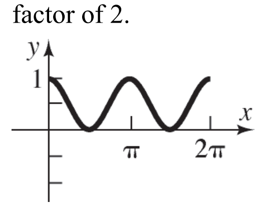
109. $f(x) = \sin^2 x = \frac{1-\cos(2x)}{2}$

Starting with the graph of $y = \cos x$, compress horizontally by a factor of 2, reflect across the x-axis, shift 1 unit up, and shrink vertically by a factor of 2.



110. $g(x) = \cos^2 x = \frac{1+\cos(2x)}{2}$

Starting with the graph of $y = \cos x$, compress horizontally by a factor of 2, reflect across the x-axis, shift 1 unit up, and shrink vertically by a factor of 2.



111. $\sin\frac{\pi}{24}=\sin\left(\frac{\frac{\pi}{12}}{2}\right)=\sqrt{\frac{1-\cos\frac{\pi}{12}}{2}}$

$=\sqrt{\frac{1-\left(\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)}{2}}=\sqrt{\frac{1}{2}-\frac{1}{8}\left(\sqrt{6}+\sqrt{2}\right)}$

$=\sqrt{\frac{8-2\left(\sqrt{6}+\sqrt{2}\right)}{16}}=\frac{\sqrt{8-2\left(\sqrt{6}+\sqrt{2}\right)}}{4}$

$=\frac{\sqrt{2\left(4-\left(\sqrt{6}+\sqrt{2}\right)\right)}}{4}=\frac{\sqrt{2}}{4}\sqrt{4-\sqrt{6}-\sqrt{2}}$

$\cos\frac{\pi}{24}=\cos\left(\frac{\frac{\pi}{12}}{2}\right)=\sqrt{\frac{1+\cos\frac{\pi}{12}}{2}}$

$=\sqrt{\frac{1+\left(\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)}{2}}=\sqrt{\frac{1}{2}+\frac{1}{8}\left(\sqrt{6}+\sqrt{2}\right)}$

$=\sqrt{\frac{8+2\left(\sqrt{6}+\sqrt{2}\right)}{16}}=\frac{\sqrt{8+2\left(\sqrt{6}+\sqrt{2}\right)}}{4}$

$=\frac{\sqrt{2\left(4+\sqrt{6}+\sqrt{2}\right)}}{4}=\frac{\sqrt{2}}{4}\sqrt{4+\sqrt{6}+\sqrt{2}}$

112. $\cos\frac{\pi}{8}=\cos\left(\frac{\frac{\pi}{4}}{2}\right)=\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}$

$=\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=\sqrt{\frac{2+\sqrt{2}}{4}}$

$=\frac{\sqrt{2+\sqrt{2}}}{2}$

$\sin\frac{\pi}{16}=\sin\left(\frac{\frac{\pi}{8}}{2}\right)=\sqrt{\frac{1-\cos\frac{\pi}{8}}{2}}$

$=\sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2}}}{2}}{2}}=\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}}$

$=\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$

$\cos\frac{\pi}{16}=\cos\left(\frac{\frac{\pi}{8}}{2}\right)=\sqrt{\frac{1+\cos\frac{\pi}{8}}{2}}$

$=\sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}}=\sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{4}}$

$=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

113. $\sin^3\theta+\sin^3(\theta+120^\circ)+\sin^3(\theta+240^\circ)$

$=\sin^3\theta+\left(\sin\theta\cos(120^\circ)+\cos\theta\sin(120^\circ)\right)^3+\left(\sin\theta\cos(240^\circ)+\cos\theta\sin(240^\circ)\right)^3$

$=\sin^3\theta+\left(-\frac{1}{2}\cdot\sin\theta+\frac{\sqrt{3}}{2}\cdot\cos\theta\right)^3+\left(-\frac{1}{2}\cdot\sin\theta-\frac{\sqrt{3}}{2}\cdot\cos\theta\right)^3$

$=\sin^3\theta+\frac{1}{8}\cdot\left(-\sin^3\theta+3\sqrt{3}\sin^2\theta\cos\theta-9\sin\theta\cos^2\theta+3\sqrt{3}\cos^3\theta\right)$

$\quad-\frac{1}{8}\left(\sin^3\theta+3\sqrt{3}\sin^2\theta\cos\theta+9\sin\theta\cos^2\theta+3\sqrt{3}\cos^3\theta\right)$

$=\sin^3\theta-\frac{1}{8}\cdot\sin^3\theta+\frac{3\sqrt{3}}{8}\cdot\sin^2\theta\cos\theta-\frac{9}{8}\cdot\sin\theta\cos^2\theta+\frac{3\sqrt{3}}{8}\cdot\cos^3\theta$

$\quad-\frac{1}{8}\cdot\sin^3\theta-\frac{3\sqrt{3}}{8}\cdot\sin^2\theta\cos\theta-\frac{9}{8}\cdot\sin\theta\cos^2\theta-\frac{3\sqrt{3}}{8}\cdot\cos^3\theta$

$=\frac{3}{4}\cdot\sin^3\theta-\frac{9}{4}\cdot\sin\theta\cos^2\theta=\frac{3}{4}\cdot\left[\sin^3\theta-3\sin\theta\left(1-\sin^2\theta\right)\right]=\frac{3}{4}\cdot\left(\sin^3\theta-3\sin\theta+3\sin^3\theta\right)$

$=\frac{3}{4}\cdot\left(4\sin^3\theta-3\sin\theta\right)=-\frac{3}{4}\cdot\sin(3\theta)$ (from Example 2)

114. $\tan\theta = \tan\left(3\cdot\frac{\theta}{3}\right)$

$$= \frac{3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3}}{1-3\tan^2\frac{\theta}{3}} \quad \text{(from problem 65)}$$

$$a\tan\frac{\theta}{3} = \frac{\tan\frac{\theta}{3}\left(3-\tan^2\frac{\theta}{3}\right)}{1-3\tan^2\frac{\theta}{3}}$$

$$3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3} = a\tan\frac{\theta}{3}\left(1-3\tan^2\frac{\theta}{3}\right)$$

$$3-\tan^2\frac{\theta}{3} = a\left(1-3\tan^2\frac{\theta}{3}\right)$$

$$3-\tan^2\frac{\theta}{3} = a - 3a\tan^2\frac{\theta}{3}$$

$$3a\tan^2\frac{\theta}{3} - \tan^2\frac{\theta}{3} = a-3$$

$$(3a-1)\tan^2\frac{\theta}{3} = a-3$$

$$\tan^2\frac{\theta}{3} = \frac{a-3}{3a-1}$$

$$\tan\frac{\theta}{3} = \pm\sqrt{\frac{a-3}{3a-1}}$$

115.
$$\cos(2x) + (2m-1)\sin x + m - 1 = 0$$
$$(1-2\sin^2 x) + (2m-1)\sin x + m - 1 = 0$$
$$-2\sin^2 x + (2m-1)\sin x + m = 0$$
$$2\sin^2 x - (2m-1)\sin x - m = 0$$

We can solve this as a quadratic equation. In order for the equation to have exactly one real solution, then $b^2 - 4ac = 0$.

$$[-(2m-1)]^2 - 4(2)(-m) = 0$$
$$4m^2 - 4m + 1 + 8m = 0$$
$$4m^2 + 4m + 1 = 0$$
$$(2m+1)^2 = 0$$

So $m = -\frac{1}{2}$.

116. Answers will vary.

117. Since the line is perpendicular the slope would be $m = \frac{1}{2}$.

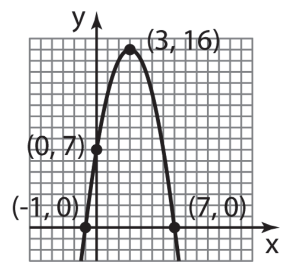
$$y - y_1 = \frac{1}{2}(x - x_1)$$
$$y - (-3) = \frac{1}{2}(x-2)$$
$$y + 3 = \frac{1}{2}x - 1$$
$$y = \frac{1}{2}x - 4$$

118. Vertex: $x = -\frac{b}{2a} = -\frac{6}{2(-1)} = 3$

$f(3) = -(3)^2 + 6(3) + 7 = 16$; $(3,16)$

x-intercepts: $0 = -x^2 + 6x + 7$
$$0 = x^2 - 6x - 7$$
$$0 = (x-7)(x+1)$$
$$x = 7 \quad or \quad x = -1$$

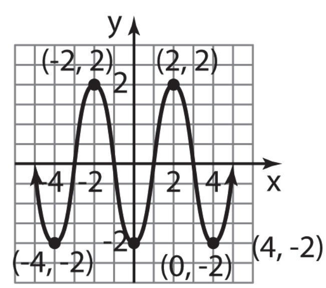
y-intercepts: $y = -(0)^2 + 6(0) + 7$
$$y = 7$$



119. $\sin\left(\frac{2\pi}{3}\right) - \cos\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2} - \left(-\frac{1}{2}\right)$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3}+1}{2}$$

120. Amplitude: 2; Period: $\frac{2\pi}{\pi/2} = 4$



Section 7.7

1. $\sin(195°)\cos(75°) = \sin(150° + 45°)\cos(30° + 45°)$

$$\sin(150° + 45°)\cos(30° + 45°) =$$
$$= \left(\sin 150° \cos 45° + \cos 150° \sin 45°\right)\left(\cos 30° \cos 45° - \sin 30° \sin 45°\right)$$
$$= \left[\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]\left[\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]$$
$$= \left(\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\right)\left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right) = \frac{\sqrt{12}}{16} - \frac{\sqrt{4}}{16} - \frac{\sqrt{36}}{16} + \frac{\sqrt{12}}{16}$$
$$= \frac{2\sqrt{3}}{16} - \frac{2}{16} - \frac{6}{16} + \frac{2\sqrt{3}}{16} = \frac{\sqrt{3}}{8} - \frac{1}{8} - \frac{3}{8} + \frac{\sqrt{3}}{8} = \frac{2\sqrt{3}}{8} - \frac{4}{8} = \frac{\sqrt{3}}{4} - \frac{1}{2} = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$$

2. $\cos(285°)\cos(195°) = \cos(240° + 45°)\cos(240° - 45°)$

$$\cos(240° + 45°)\cos(240° - 45°) =$$
$$= \left(\cos 240° \cos 45° - \sin 240° \sin 45°\right)\left(\cos 240° \cos 45° + \sin 240° \sin 45°\right)$$
$$= \left(\cos 240°\right)^2 \left(\cos 45°\right)^2 - \left(\sin 240°\right)^2 \left(\sin 45°\right)^2$$
$$= \left(-\frac{1}{2}\right)^2 \left(\frac{\sqrt{2}}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2 \left(\frac{\sqrt{2}}{2}\right)^2 = \left(\frac{1}{4}\right)\left(\frac{2}{4}\right) - \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)$$
$$= \frac{1}{8} - \frac{3}{8} = -\frac{1}{4}$$

3. $\sin(285°)\sin(75°) = \sin(240° + 45°)\sin(30° + 45°)$

$$\sin(240° + 45°)\sin(30° + 45°) =$$
$$= \left(\sin 240° \cos 45° + \cos 240° \sin 45°\right)\left(\sin 30° \cos 45° + \cos 30° \sin 45°\right)$$
$$= \left[\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]\left[\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]$$
$$= \left(-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right)\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\right) = -\frac{\sqrt{12}}{16} - \frac{\sqrt{36}}{16} - \frac{\sqrt{4}}{16} - \frac{\sqrt{12}}{16}$$
$$= -\frac{2\sqrt{3}}{16} - \frac{6}{16} - \frac{2}{16} - \frac{2\sqrt{3}}{16} = -\frac{\sqrt{3}}{8} - \frac{3}{8} - \frac{1}{8} - \frac{\sqrt{3}}{8} = -\frac{2\sqrt{3}}{8} - \frac{4}{8} = -\frac{\sqrt{3}}{4} - \frac{1}{2} = -\frac{1}{2}\left(\frac{\sqrt{3}}{2} + 1\right)$$

4. $$\begin{aligned}\sin(75°)+\sin(15°) &= \sin(45°+30°)+\sin(45°-30°)\\ &= \left[\sin(45°)\cos(30°)+\cos(45°)\sin(30°)\right]+\left[\sin(45°)\cos(30°)-\cos(45°)\sin(30°)\right]\\ &= 2\sin(45°)\cos(30°)\\ &= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{6}}{2}\end{aligned}$$

5. $$\begin{aligned}\cos(255°)-\cos(195°) &= \cos(225°+30°)-\cos(225°-30°)\\ &= \left[\cos(225°)\cos(30°)-\sin(225°)\sin(30°)\right]-\left[\cos(225°)\cos(30°)+\sin(225°)\sin(30°)\right]\\ &= -2\sin(225°)\sin(30°)\\ &= -2\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)=\frac{\sqrt{2}}{2}\end{aligned}$$

6. $$\begin{aligned}\sin(255°)-\sin(15°) &= \sin(135°+120°)-\sin(135°-120°)\\ &= \left[\sin(135°)\cos(120°)+\cos(135°)\sin(120°)\right]-\left[\sin(135°)\cos(120°)-\cos(135°)\sin(120°)\right]\\ &= \sin(135°)\cos(120°)+\cos(135°)\sin(120°)-\sin(135°)\cos(120°)+\cos(135°)\sin(120°)\\ &= 2\cos(135°)\sin(120°)\\ &= 2\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)=-\frac{\sqrt{6}}{2}\end{aligned}$$

7. $$\begin{aligned}\sin(4\theta)\sin(2\theta) &= \frac{1}{2}\left[\cos(4\theta-2\theta)-\cos(4\theta+2\theta)\right]\\ &= \frac{1}{2}\left[\cos(2\theta)-\cos(6\theta)\right]\end{aligned}$$

8. $$\begin{aligned}\cos(4\theta)\cos(2\theta) &= \frac{1}{2}\left[\cos(4\theta-2\theta)+\cos(4\theta+2\theta)\right]\\ &= \frac{1}{2}\left[\cos(2\theta)+\cos(6\theta)\right]\end{aligned}$$

9. $$\begin{aligned}\sin(4\theta)\cos(2\theta) &= \frac{1}{2}\left[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)\right]\\ &= \frac{1}{2}\left[\sin(6\theta)+\sin(2\theta)\right]\end{aligned}$$

10. $$\begin{aligned}\sin(3\theta)\sin(5\theta) &= \frac{1}{2}\left[\cos(3\theta-5\theta)-\cos(3\theta+5\theta)\right]\\ &= \frac{1}{2}\left[\cos(-2\theta)-\cos(8\theta)\right]\\ &= \frac{1}{2}\left[\cos(2\theta)-\cos(8\theta)\right]\end{aligned}$$

11. $$\begin{aligned}\cos(3\theta)\cos(5\theta) &= \frac{1}{2}\left[\cos(3\theta-5\theta)+\cos(3\theta+5\theta)\right]\\ &= \frac{1}{2}\left[\cos(-2\theta)+\cos(8\theta)\right]\\ &= \frac{1}{2}\left[\cos(2\theta)+\cos(8\theta)\right]\end{aligned}$$

12. $$\begin{aligned}\sin(4\theta)\cos(6\theta) &= \frac{1}{2}\left[\sin(4\theta+6\theta)+\sin(4\theta-6\theta)\right]\\ &= \frac{1}{2}\left[\sin(10\theta)+\sin(-2\theta)\right]\\ &= \frac{1}{2}\left[\sin(10\theta)-\sin(2\theta)\right]\end{aligned}$$

13. $$\begin{aligned}\sin\theta\sin(2\theta) &= \frac{1}{2}\left[\cos(\theta-2\theta)-\cos(\theta+2\theta)\right]\\ &= \frac{1}{2}\left[\cos(-\theta)-\cos(3\theta)\right]\\ &= \frac{1}{2}\left[\cos\theta-\cos(3\theta)\right]\end{aligned}$$

14. $\cos(3\theta)\cos(4\theta) = \frac{1}{2}[\cos(3\theta - 4\theta) + \cos(3\theta + 4\theta)]$
$= \frac{1}{2}[\cos(-\theta) + \cos(7\theta)]$
$= \frac{1}{2}[\cos\theta + \cos(7\theta)]$

15. $\sin\frac{3\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\left[\sin\left(\frac{3\theta}{2} + \frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2} - \frac{\theta}{2}\right)\right]$
$= \frac{1}{2}[\sin(2\theta) + \sin\theta]$

16. $\sin\frac{\theta}{2}\cos\frac{5\theta}{2} = \frac{1}{2}\left[\sin\left(\frac{\theta}{2} + \frac{5\theta}{2}\right) + \sin\left(\frac{\theta}{2} - \frac{5\theta}{2}\right)\right]$
$= \frac{1}{2}[\sin(3\theta) + \sin(-2\theta)]$
$= \frac{1}{2}[\sin(3\theta) - \sin(2\theta)]$

17. $\sin(4\theta) - \sin(2\theta) = 2\sin\left(\frac{4\theta - 2\theta}{2}\right)\cos\left(\frac{4\theta + 2\theta}{2}\right)$
$= 2\sin\theta\cos(3\theta)$

18. $\sin(4\theta) + \sin(2\theta) = 2\sin\left(\frac{4\theta + 2\theta}{2}\right)\cos\left(\frac{4\theta - 2\theta}{2}\right)$
$= 2\sin(3\theta)\cos\theta$

19. $\cos(2\theta) + \cos(4\theta) = 2\cos\left(\frac{2\theta + 4\theta}{2}\right)\cos\left(\frac{2\theta - 4\theta}{2}\right)$
$= 2\cos(3\theta)\cos(-\theta)$
$= 2\cos(3\theta)\cos\theta$

20. $\cos(5\theta) - \cos(3\theta) = -2\sin\left(\frac{5\theta + 3\theta}{2}\right)\sin\left(\frac{5\theta - 3\theta}{2}\right)$
$= -2\sin(4\theta)\sin\theta$

21. $\sin\theta + \sin(3\theta) = 2\sin\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right)$
$= 2\sin(2\theta)\cos(-\theta)$
$= 2\sin(2\theta)\cos\theta$

22. $\cos\theta + \cos(3\theta) = 2\cos\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right)$
$= 2\cos(2\theta)\cos(-\theta)$
$= 2\cos(2\theta)\cos\theta$

23. $\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} = -2\sin\left(\frac{\frac{\theta}{2} + \frac{3\theta}{2}}{2}\right)\sin\left(\frac{\frac{\theta}{2} - \frac{3\theta}{2}}{2}\right)$
$= -2\sin\theta\sin\left(-\frac{\theta}{2}\right)$
$= -2\sin\theta\left(-\sin\frac{\theta}{2}\right)$
$= 2\sin\theta\sin\frac{\theta}{2}$

24. $\sin\frac{\theta}{2} - \sin\frac{3\theta}{2} = 2\sin\left(\frac{\frac{\theta}{2} - \frac{3\theta}{2}}{2}\right)\cos\left(\frac{\frac{\theta}{2} + \frac{3\theta}{2}}{2}\right)$
$= 2\sin\left(-\frac{\theta}{2}\right)\cos\theta$
$= -2\sin\frac{\theta}{2}\cos\theta$

25. $\frac{\sin\theta + \sin(3\theta)}{2\sin(2\theta)} = \frac{2\sin\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right)}{2\sin(2\theta)}$
$= \frac{2\sin(2\theta)\cos(-\theta)}{2\sin(2\theta)}$
$= \cos(-\theta)$
$= \cos\theta$

26. $\frac{\cos\theta + \cos(3\theta)}{2\cos(2\theta)} = \frac{2\cos\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right)}{2\cos(2\theta)}$
$= \frac{2\cos(2\theta)\cos(-\theta)}{2\cos(2\theta)}$
$= \cos(-\theta)$
$= \cos\theta$

27. $$\frac{\sin(4\theta)+\sin(2\theta)}{\cos(4\theta)+\cos(2\theta)}=\frac{2\sin\left(\frac{4\theta+2\theta}{2}\right)\cos\left(\frac{4\theta-2\theta}{2}\right)}{\cos(4\theta)+\cos(2\theta)}$$
$$=\frac{2\sin(3\theta)\cos\theta}{2\cos(3\theta)\cos\theta}$$
$$=\frac{\sin(3\theta)}{\cos(3\theta)}$$
$$=\tan(3\theta)$$

28. $$\frac{\cos\theta-\cos(3\theta)}{\sin(3\theta)-\sin\theta}=\frac{-2\sin\left(\frac{\theta+3\theta}{2}\right)\sin\left(\frac{\theta-3\theta}{2}\right)}{2\sin\left(\frac{3\theta-\theta}{2}\right)\cos\left(\frac{3\theta+\theta}{2}\right)}$$
$$=\frac{-2\sin(2\theta)\sin(-\theta)}{2\sin\theta\cos(2\theta)}$$
$$=\frac{-(-\sin\theta)\sin(2\theta)}{\sin\theta\cos(2\theta)}$$
$$=\tan(2\theta)$$

29. $$\frac{\cos\theta-\cos(3\theta)}{\sin\theta+\sin(3\theta)}=\frac{-2\sin\left(\frac{\theta+3\theta}{2}\right)\sin\left(\frac{\theta-3\theta}{2}\right)}{2\sin\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)}$$
$$=\frac{-2\sin(2\theta)\sin(-\theta)}{2\sin(2\theta)\cos(-\theta)}$$
$$=\frac{-(-\sin\theta)}{\cos\theta}$$
$$=\tan\theta$$

30. $$\frac{\cos\theta-\cos(5\theta)}{\sin\theta+\sin(5\theta)}=\frac{-2\sin\left(\frac{\theta+5\theta}{2}\right)\sin\left(\frac{\theta-5\theta}{2}\right)}{2\sin\left(\frac{\theta+5\theta}{2}\right)\cos\left(\frac{\theta-5\theta}{2}\right)}$$
$$=\frac{-2\sin(3\theta)\sin(-2\theta)}{2\sin(3\theta)\cos(-2\theta)}$$
$$=\frac{-(-\sin 2\theta)}{\cos(2\theta)}$$
$$=\tan(2\theta)$$

31. $$\sin\theta[\sin\theta+\sin(3\theta)]$$
$$=\sin\theta\left[2\sin\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)\right]$$
$$=\sin\theta[2\sin(2\theta)\cos(-\theta)]$$
$$=\cos\theta[2\sin(2\theta)\sin\theta]$$
$$=\cos\theta\left[2\cdot\frac{1}{2}[\cos\theta-\cos(3\theta)]\right]$$
$$=\cos\theta[\cos\theta-\cos(3\theta)]$$

32. $$\sin\theta[\sin(3\theta)+\sin(5\theta)]$$
$$=\sin\theta\left[2\sin\left(\frac{3\theta+5\theta}{2}\right)\cos\left(\frac{3\theta-5\theta}{2}\right)\right]$$
$$=\sin\theta[2\sin(4\theta)\cos(-\theta)]$$
$$=\cos\theta[2\sin(4\theta)\sin\theta]$$
$$=\cos\theta\left[2\cdot\frac{1}{2}[\cos(3\theta)-\cos(5\theta)]\right]$$
$$=\cos\theta[\cos(3\theta)-\cos(5\theta)]$$

33. $$\frac{\sin(4\theta)+\sin(8\theta)}{\cos(4\theta)+\cos(8\theta)}$$
$$=\frac{2\sin\left(\frac{4\theta+8\theta}{2}\right)\cos\left(\frac{4\theta-8\theta}{2}\right)}{2\cos\left(\frac{4\theta+8\theta}{2}\right)\cos\left(\frac{4\theta-8\theta}{2}\right)}$$
$$=\frac{2\sin(6\theta)\cos(-2\theta)}{2\cos(6\theta)\cos(-2\theta)}$$
$$=\frac{\sin(6\theta)}{\cos(6\theta)}$$
$$=\tan(6\theta)$$

34. $$\frac{\sin(4\theta)-\sin(8\theta)}{\cos(4\theta)-\cos(8\theta)}$$
$$=\frac{2\sin\left(\frac{4\theta-8\theta}{2}\right)\cos\left(\frac{4\theta+8\theta}{2}\right)}{-2\sin\left(\frac{4\theta+8\theta}{2}\right)\sin\left(\frac{4\theta-8\theta}{2}\right)}$$
$$=\frac{2\sin(-2\theta)\cos(6\theta)}{-2\sin(6\theta)\sin(-2\theta)}$$
$$=\frac{\cos(6\theta)}{-\sin(6\theta)}$$
$$=-\cot(6\theta)$$

35. $\dfrac{\sin(4\theta)+\sin(8\theta)}{\sin(4\theta)-\sin(8\theta)}$

$$=\frac{2\sin\left(\frac{4\theta+8\theta}{2}\right)\cos\left(\frac{4\theta-8\theta}{2}\right)}{-2\sin\left(\frac{4\theta-8\theta}{2}\right)\cos\left(\frac{4\theta+8\theta}{2}\right)}$$

$$=\frac{2\sin(6\theta)\cos(-2\theta)}{2\sin(-2\theta)\cos(6\theta)}$$

$$=\frac{\sin(6\theta)\cos(2\theta)}{-\sin(2\theta)\cos(6\theta)}$$

$$=-\tan(6\theta)\cot(2\theta)$$

$$=-\frac{\tan(6\theta)}{\tan(2\theta)}$$

36. $\dfrac{\cos(4\theta)-\cos(8\theta)}{\cos(4\theta)+\cos(8\theta)}$

$$=\frac{-2\sin\left(\frac{4\theta+8\theta}{2}\right)\sin\left(\frac{4\theta-8\theta}{2}\right)}{2\cos\left(\frac{4\theta+8\theta}{2}\right)\cos\left(\frac{4\theta-8\theta}{2}\right)}$$

$$=\frac{-2\sin(6\theta)\sin(-2\theta)}{2\cos(6\theta)\cos(-2\theta)}$$

$$=-\frac{\sin(6\theta)}{\cos(6\theta)}\cdot\frac{\sin(-2\theta)}{\cos(-2\theta)}$$

$$=-\tan(6\theta)\tan(-2\theta)$$

$$=\tan(2\theta)\tan(6\theta)$$

37. $$\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$=\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$=\tan\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$$

38. $$\frac{\cos\alpha+\cos\beta}{\cos\alpha-\cos\beta}=\frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$=-\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$=-\cot\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$$

39. $$\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$=\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$=\tan\left(\frac{\alpha+\beta}{2}\right)$$

40. $$\frac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta}=\frac{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$=-\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}$$

$$=-\cot\left(\frac{\alpha+\beta}{2}\right)$$

41. $$\begin{aligned} 1+\cos(2\theta)+\cos(4\theta)+\cos(6\theta) &= \cos 0+\cos(6\theta)+\cos(2\theta)+\cos(4\theta) \\ &= 2\cos\left(\frac{0+6\theta}{2}\right)\cos\left(\frac{0-6\theta}{2}\right)+2\cos\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta-4\theta}{2}\right) \\ &= 2\cos(3\theta)\cos(-3\theta)+2\cos(3\theta)\cos(-\theta) \\ &= 2\cos^2(3\theta)+2\cos(3\theta)\cos\theta \\ &= 2\cos(3\theta)\left[\cos(3\theta)+\cos\theta\right] \\ &= 2\cos(3\theta)\left[2\cos\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)\right] \\ &= 2\cos(3\theta)\left[2\cos(2\theta)\cos\theta\right] \\ &= 4\cos\theta\cos(2\theta)\cos(3\theta) \end{aligned}$$

42. $$\begin{aligned} 1-\cos(2\theta)+\cos(4\theta)-\cos(6\theta) &= \left[\cos 0-\cos(6\theta)\right]+\left[\cos(4\theta)-\cos(2\theta)\right] \\ &= -2\sin\left(\frac{0+6\theta}{2}\right)\sin\left(\frac{0-6\theta}{2}\right)-2\sin\left(\frac{2\theta+4\theta}{2}\right)\sin\left(\frac{2\theta-4\theta}{2}\right) \\ &= -2\sin(3\theta)\sin(-3\theta)-2\sin(3\theta)\sin(\theta) \\ &= 2\sin^2(3\theta)-2\sin(3\theta)\sin\theta \\ &= 2\sin(3\theta)\left[\sin(3\theta)-\sin\theta\right] \\ &= 2\sin(3\theta)\left[2\sin\left(\frac{3\theta-\theta}{2}\right)\cos\left(\frac{3\theta+\theta}{2}\right)\right] \\ &= 2\sin(3\theta)\left[2\sin\theta\cos(2\theta)\right] \\ &= 4\sin\theta\cos(2\theta)\sin(3\theta) \end{aligned}$$

43. $$\begin{aligned} \sin(2\theta)+\sin(4\theta) &= 0 \\ \sin(2\theta)+2\sin(2\theta)\cos(2\theta) &= 0 \\ \sin(2\theta)\left(1+2\cos(2\theta)\right) &= 0 \end{aligned}$$

$\sin(2\theta)=0$ or $1+2\cos(2\theta)=0$

$\cos(2\theta)=-\frac{1}{2}$

$2\theta=0+2k\pi$ or $2\theta=\pi+2k\pi$ or

$\theta=k\pi$ $\qquad \theta=\frac{\pi}{2}+k\pi$

$2\theta=\frac{2\pi}{3}+2k\pi$ or $2\theta=\frac{4\pi}{3}+2k\pi$

$\theta=\frac{\pi}{3}+k\pi$ $\qquad \theta=\frac{2\pi}{3}+k\pi$

On the interval $0\le\theta<2\pi$, the solution set is $\left\{0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}\right\}$.

44. $$\begin{aligned} \cos(2\theta)+\cos(4\theta) &= 0 \\ 2\cos\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta-4\theta}{2}\right) &= 0 \\ 2\cos(3\theta)\cos(-\theta) &= 0 \\ 2\cos(3\theta)\cos\theta &= 0 \end{aligned}$$

$\cos(3\theta)=0$ or $\cos\theta=0$

$3\theta=\frac{\pi}{2}+2k\pi$ or $3\theta=\frac{3\pi}{2}+2k\pi$ or

$\theta=\frac{\pi}{6}+\frac{2k\pi}{3}$ $\qquad \theta=\frac{\pi}{2}+\frac{2k\pi}{3}$

$\theta=\frac{\pi}{2}+2k\pi$ or $\theta=\frac{3\pi}{2}+2k\pi$

On the interval $0\le\theta<2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$.

45.
$$\cos(4\theta) - \cos(6\theta) = 0$$
$$-2\sin\left(\frac{4\theta + 6\theta}{2}\right)\sin\left(\frac{4\theta - 6\theta}{2}\right) = 0$$
$$-2\sin(5\theta)\sin(-\theta) = 0$$
$$2\sin(5\theta)\sin\theta = 0$$
$$\sin(5\theta) = 0 \quad \text{or} \quad \sin\theta = 0$$
$$5\theta = 0 + 2k\pi \quad \text{or} \quad 5\theta = \pi + 2k\pi \quad \text{or}$$
$$\theta = \frac{2k\pi}{5} \qquad \theta = \frac{\pi}{5} + \frac{2k\pi}{5}$$
$$\theta = 0 + 2k\pi \quad \text{or} \quad \theta = \pi + 2k\pi$$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}\right\}$.

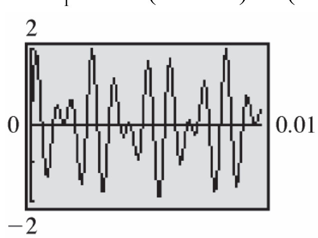
46.
$$\sin(4\theta) - \sin(6\theta) = 0$$
$$2\sin\left(\frac{4\theta - 6\theta}{2}\right)\cos\left(\frac{4\theta + 6\theta}{2}\right) = 0$$
$$2\sin(-\theta)\cos(5\theta) = 0$$
$$-2\sin\theta\cos(5\theta) = 0$$
$$\cos(5\theta) = 0 \quad \text{or} \quad \sin\theta = 0$$
$$\theta = 0 + 2k\pi \quad \text{or} \quad \theta = \pi + 2k\pi \quad \text{or}$$
$$5\theta = \frac{\pi}{2} + 2k\pi \quad \text{or} \quad 5\theta = \frac{3\pi}{2} + 2k\pi$$
$$\theta = \frac{\pi}{10} + \frac{2k\pi}{5} \qquad \theta = \frac{3\pi}{10} + \frac{2k\pi}{5}$$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{0, \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}, \pi, \frac{11\pi}{10}, \frac{13\pi}{10}, \frac{3\pi}{2}, \frac{17\pi}{10}, \frac{19\pi}{10}\right\}$.

47. a.
$$y = \sin\left[2\pi(852)t\right] + \sin\left[2\pi(1209)t\right]$$
$$= 2\sin\left(\frac{2\pi(852)t + 2\pi(1209)t}{2}\right)\cos\left(\frac{2\pi(852)t - 2\pi(1209)t}{2}\right)$$
$$= 2\sin(2061\pi t)\cos(-357\pi t)$$
$$= 2\sin(2061\pi t)\cos(357\pi t)$$

b. Because $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$ for all θ, it follows that $|\sin(2061\pi t)| \le 1$ and $|\cos(357\pi t)| \le 1$ for all values of *t*. Thus, $y = 2\sin(2061\pi t)\cos(357\pi t) \le 2 \cdot 1 \cdot 1 = 2$. That is, the maximum value of *y* is 2.

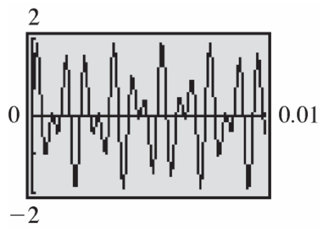
c. Let $Y_1 = 2\sin(2061\pi x)\cos(357\pi x)$.



48. a.
$$y = \sin\left[2\pi(941)t\right] + \sin\left[2\pi(1477)t\right]$$
$$= 2\sin\left(\frac{2\pi(941)t + 2\pi(1477)t}{2}\right)\cos\left(\frac{2\pi(941)t - 2\pi(1477)t}{2}\right)$$
$$= 2\sin(2418\pi t)\cos(-536\pi t)$$
$$= 2\sin(2418\pi t)\cos(536\pi t)$$

b. Because $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$ for all θ, it follows that $|\sin(2418\pi t)| \le 1$ and $|\cos(2418\pi t)| \le 1$ for all values of *t*. Thus, $y = 2\sin(2418\pi t)\cos(536\pi t) \le 2 \cdot 1 \cdot 1 = 2$. That is, the maximum value of *y* is 2.

c. Let $Y_1 = 2\sin(2418\pi x)\cos(536\pi x)$.



49. $$I_u = I_x \cos^2\theta + I_y \sin^2\theta - 2I_{xy}\sin\theta\cos\theta$$
$$= I_x\left(\frac{\cos 2\theta + 1}{2}\right) + I_y\left(\frac{1-\cos 2\theta}{2}\right) - I_{xy}2\sin\theta\cos\theta$$
$$= \frac{I_x\cos 2\theta}{2} + \frac{I_x}{2} + \frac{I_y}{2} - \frac{I_y\cos 2\theta}{2} - I_{xy}\sin 2\theta$$
$$= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_v = I_x\sin^2\theta + I_y\cos^2\theta + 2I_{xy}\sin\theta\cos\theta$$
$$= I_x\left(\frac{1-\cos 2\theta}{2}\right) + I_y\left(\frac{\cos 2\theta + 1}{2}\right) + I_{xy}2\sin\theta\cos\theta$$
$$= \frac{I_x}{2} - \frac{I_x\cos 2\theta}{2} + \frac{I_y\cos 2\theta}{2} + \frac{I_y}{2} + I_{xy}\sin 2\theta$$
$$= \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

50. **a.** Since ϕ and v_0 are fixed, we need to maximize $\sin\theta\cos(\theta-\phi)$.

$$\sin\theta\cos(\theta-\phi) = \frac{1}{2}\left[\sin\left(\theta+(\theta-\phi)\right) + \sin\left(\theta-(\theta-\phi)\right)\right]$$
$$= \frac{1}{2}\left[\sin(2\theta-\phi) + \sin\phi\right]$$

This quantity will be maximized when $\sin(2\theta-\phi) = 1$. So,

$$R_{\max} = \frac{2v_0^2\cdot\frac{1}{2}\cdot(1+\sin\phi)}{g\cos^2\phi} = \frac{v_0^2(1+\sin\phi)}{g(1-\sin^2\phi)} = \frac{v_0^2(1+\sin\phi)}{g(1-\sin\phi)(1+\sin\phi)} = \frac{v_0^2}{g(1-\sin\phi)}$$

b. $$R_{\max} = \frac{(50)^2}{9.8(1-\sin 35°)} \approx 598.24$$

The maximum range is about 598 meters.

51. $\sin(2\alpha)+\sin(2\beta)+\sin(2\gamma)$

$$
\begin{aligned}
&= 2\sin\left(\frac{2\alpha+2\beta}{2}\right)\cos\left(\frac{2\alpha-2\beta}{2}\right)+\sin(2\gamma)\\
&= 2\sin(\alpha+\beta)\cos(\alpha-\beta)+2\sin\gamma\cos\gamma\\
&= 2\sin(\pi-\gamma)\cos(\alpha-\beta)+2\sin\gamma\cos\gamma\\
&= 2\sin\gamma\cos(\alpha-\beta)+2\sin\gamma\cos\gamma\\
&= 2\sin\gamma\left[\cos(\alpha-\beta)+\cos\gamma\right]\\
&= 2\sin\gamma\left[2\cos\left(\frac{\alpha-\beta+\gamma}{2}\right)\cos\left(\frac{\alpha-\beta-\gamma}{2}\right)\right]\\
&= 4\sin\gamma\cos\left(\frac{\pi-2\beta}{2}\right)\cos\left(\frac{2\alpha-\pi}{2}\right)\\
&= 4\sin\gamma\cos\left(\frac{\pi}{2}-\beta\right)\cos\left(\alpha-\frac{\pi}{2}\right)\\
&= 4\sin\gamma\sin\beta\sin\alpha\\
&= 4\sin\alpha\sin\beta\sin\gamma
\end{aligned}
$$

52. $\tan\alpha+\tan\beta+\tan\gamma=\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}+\dfrac{\sin\gamma}{\cos\gamma}$

$$
\begin{aligned}
&= \frac{\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\alpha\cos\gamma+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\cos\gamma(\sin\alpha\cos\beta+\cos\alpha\sin\beta)+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\cos\gamma\sin(\alpha+\beta)+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}=\frac{\cos\gamma\sin(\pi-\gamma)+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\cos\gamma\sin\gamma+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}=\frac{\sin\gamma(\cos\gamma+\cos\alpha\cos\beta)}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\sin\gamma\left[\cos\left(\pi-(\alpha+\beta)\right)+\cos\alpha\cos\beta\right]}{\cos\alpha\cos\beta\cos\gamma}=\frac{\sin\gamma\left[-\cos(\alpha+\beta)+\cos\alpha\cos\beta\right]}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\sin\gamma\left(-\cos\alpha\cos\beta+\sin\alpha\sin\beta+\cos\alpha\cos\beta\right)}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\sin\gamma(\sin\alpha\sin\beta)}{\cos\alpha\cos\beta\cos\gamma}=\tan\alpha\tan\beta\tan\gamma
\end{aligned}
$$

53. Add the sum formulas for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$ and solve for $\sin\alpha\,\cos\beta$:

$$
\begin{aligned}
\sin(\alpha+\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\
\sin(\alpha-\beta)&=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\
\sin(\alpha+\beta)+\sin(\alpha-\beta)&=2\sin\alpha\cos\beta\\
\sin\alpha\cos\beta&=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]
\end{aligned}
$$

54. $2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

$= 2\cdot\frac{1}{2}\left[\sin\left(\frac{\alpha-\beta}{2}+\frac{\alpha+\beta}{2}\right)+\sin\left(\frac{\alpha-\beta}{2}-\frac{\alpha+\beta}{2}\right)\right]$

$= \sin\left(\frac{2\alpha}{2}\right)+\sin\left(\frac{-2\beta}{2}\right)$

$= \sin\alpha+\sin(-\beta)$

$= \sin\alpha-\sin\beta$

Thus, $\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$.

55. $2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$= 2\cdot\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\right]$

$= \cos\left(\frac{2\beta}{2}\right)+\cos\left(\frac{2\alpha}{2}\right)$

$= \cos\beta+\cos\alpha$

Thus, $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

56. $-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$= -2\cdot\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\right]$

$= -\left[\cos\left(\frac{2\beta}{2}\right)-\cos\left(\frac{2\alpha}{2}\right)\right]$

$= \cos\alpha-\cos\beta$

Thus, $\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$.

57. $27^{x-1} = 9^{x+5}$

$3^{3(x-1)} = 3^{2(x+5)}$

$3(x-1) = 2(x+5)$

$3x-3 = 2x+10$

$x = 13$

The solution set is $\{13\}$.

58. Amplitude: 5

Period: $\frac{2\pi}{4} = \frac{\pi}{2}$

Phase Shift: $-\frac{-\pi}{4} = \frac{\pi}{4}$

59. $\cos\left(\csc^{-1}\frac{7}{5}\right)$

Since $\csc\theta = \frac{7}{5},\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, let $r = 7$ and $y = 5$.

Solve for x: $x^2+25 = 49$

$x^2 = 24$

$x = \pm 2\sqrt{6}$

Since θ is in quadrant I, $x = 2\sqrt{6}$.

Thus, $\cos\left(\csc^{-1}\frac{7}{5}\right) = \cos\theta = \frac{x}{r} = \frac{2\sqrt{6}}{7}$.

60. We find the inverse function by switching the x and y variables and solving for y.

$f(x) = 3\sin x - 5$

$y = 3\sin x - 5$

$x = 3\sin y - 5$

$x+5 = 3\sin y$

$\frac{x+5}{3} = \sin y$

$\sin^{-1}\left(\frac{x+5}{3}\right) = y$

$f^{-1}(x) = \sin^{-1}\left(\frac{x+5}{3}\right)$

The domain of $\sin^{-1}(u)$ is $[-1,1]$ so

$-1 \le \left(\frac{x+5}{3}\right) \le 1$

$-8 \le x \le -2$

Range of f = Domain of $f^{-1} = [-8,-2]$

Range of f^{-1} = $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Chapter 7 Review Exercises

1. $\sin^{-1} 1$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals 1.

$\sin\theta = 1, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\theta = \frac{\pi}{2}$

Thus, $\sin^{-1}(1) = \frac{\pi}{2}$.

2. $\cos^{-1} 0$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals 0.

$\cos\theta = 0, \quad 0 \le \theta \le \pi$

$\theta = \frac{\pi}{2}$

Thus, $\cos^{-1}(0) = \frac{\pi}{2}$.

3. $\tan^{-1} 1$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 1.

$\tan\theta = 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

Thus, $\tan^{-1}(1) = \frac{\pi}{4}$.

4. $\sin^{-1}\left(-\frac{1}{2}\right)$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$\sin\theta = -\frac{1}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\theta = -\frac{\pi}{6}$

Thus, $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

5. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

$\cos\theta = -\frac{\sqrt{3}}{2}, \quad 0 \le \theta \le \pi$

$\theta = \frac{5\pi}{6}$

Thus, $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

6. $\tan^{-1}\left(-\sqrt{3}\right)$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $-\sqrt{3}$.

$\tan\theta = -\sqrt{3}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\theta = -\frac{\pi}{3}$

Thus, $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$.

7. $\sec^{-1}\sqrt{2}$

Find the angle θ, $0 \le \theta \le \pi$, whose secant equals $\sqrt{2}$.

$\sec\theta = \sqrt{2}, \quad 0 \le \theta \le \pi$

$\theta = \frac{\pi}{4}$

Thus, $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$.

8. $\cot^{-1}(-1)$

Find the angle θ, $0 < \theta < \pi$, whose cotangent equals -1.

$\cot\theta = -1, \quad 0 < \theta < \pi$

$\theta = \frac{3\pi}{4}$

Thus, $\cot^{-1}(-1) = \frac{3\pi}{4}$.

9. $\sin^{-1}\left(\sin\left(\frac{3\pi}{8}\right)\right)$ follows the form of the equation $f^{-1}(f(x))=\sin^{-1}(\sin(x))=x$. Since $\frac{3\pi}{8}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation directly and get $\sin^{-1}\left(\sin\left(\frac{3\pi}{8}\right)\right)=\frac{3\pi}{8}$.

10. $\cos^{-1}\left(\cos\frac{3\pi}{4}\right)$ follows the form of the equation $f^{-1}(f(x))=\cos^{-1}(\cos(x))=x$. Since $\frac{3\pi}{4}$ is in the interval $[0,\pi]$, we can apply the equation directly and get $\cos^{-1}\left(\cos\frac{3\pi}{4}\right)=\frac{3\pi}{4}$.

11. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$ follows the form of the equation $f^{-1}(f(x))=\tan^{-1}(\tan(x))=x$ but we cannot use the formula directly since $\frac{2\pi}{3}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\tan\left(\frac{2\pi}{3}\right)=\tan\theta$. The angle $\frac{2\pi}{3}$ is in quadrant II so tangent is negative. The reference angle of $\frac{2\pi}{3}$ is $\frac{\pi}{3}$ and we want θ to be in quadrant IV so tangent will still be negative. Thus, we have $\tan\left(\frac{2\pi}{3}\right)=\tan\left(-\frac{\pi}{3}\right)$. Since $-\frac{\pi}{3}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get

$$\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)=\tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right)=-\frac{\pi}{3}.$$

12. $\cos^{-1}\left(\cos\left(\frac{15\pi}{7}\right)\right)$ follows the form of the equation $f^{-1}(f(x))=\cos^{-1}(\cos(x))=x$, but we cannot use the formula directly since $\frac{15\pi}{7}$ is not in the interval $[0,\pi]$. We need to find an angle θ in the interval $[0,\pi]$ for which $\cos\left(\frac{15\pi}{7}\right)=\cos\theta$. The angle $\frac{15\pi}{7}$ is in quadrant I so the reference angle of $\frac{15\pi}{7}$ is $\frac{\pi}{7}$. Thus, we have $\cos\left(\frac{15\pi}{7}\right)=\cos\frac{\pi}{7}$. Since $\frac{\pi}{7}$ is in the interval $[0,\pi]$, we can apply the equation above and get

$$\cos^{-1}\left(\cos\left(\frac{15\pi}{7}\right)\right)=\cos^{-1}\left(\cos\frac{\pi}{7}\right)=\frac{\pi}{7}.$$

13. $\sin^{-1}\left(\sin\left(-\frac{8\pi}{9}\right)\right)$ follows the form of the equation $f^{-1}(f(x))=\sin^{-1}(\sin(x))=x$, but we cannot use the formula directly since $-\frac{8\pi}{9}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\sin\left(-\frac{8\pi}{9}\right)=\sin\theta$. The angle $-\frac{8\pi}{9}$ is in quadrant III so sine is negative. The reference angle of $-\frac{8\pi}{9}$ is $\frac{\pi}{9}$ and we want θ to be in quadrant IV so sine will still be negative. Thus, we have $\sin\left(-\frac{8\pi}{9}\right)=\sin\left(-\frac{\pi}{9}\right)$. Since $-\frac{\pi}{9}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get

$$\sin^{-1}\left(\sin\left(-\frac{8\pi}{9}\right)\right)=\sin^{-1}\left(\sin\left(-\frac{\pi}{9}\right)\right)=-\frac{\pi}{9}.$$

14. $\sin\left(\sin^{-1} 0.9\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\sin\left(\sin^{-1}(x)\right)=x$. Since 0.9 is in the interval $[-1,1]$, we can apply the equation directly and get $\sin\left(\sin^{-1} 0.9\right)=0.9$.

15. $\cos\left(\cos^{-1} 0.6\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\cos\left(\cos^{-1}(x)\right)=x$. Since 0.6 is in the interval $[-1,1]$, we can apply the equation directly and get $\cos\left(\cos^{-1} 0.6\right)=0.6$.

16. $\tan\left(\tan^{-1} 5\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\tan\left(\tan^{-1}(x)\right)=x$. Since 5 is a real number, we can apply the equation directly and get $\tan\left(\tan^{-1} 5\right)=5$.

17. Since there is no angle θ such that $\cos\theta=-1.6$, the quantity $\cos^{-1}(-1.6)$ is not defined. Thus, $\cos\left(\cos^{-1}(-1.6)\right)$ is not defined.

18. $\sin^{-1}\left(\cos\frac{2\pi}{3}\right)=\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

19. $\cos^{-1}\left(\tan\frac{3\pi}{4}\right)=\cos^{-1}(-1)=\pi$

20. $\tan\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta=-\frac{\sqrt{3}}{2}, \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$
$$\theta=-\frac{\pi}{3}$$

So, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$.

Thus, $\tan\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]=\tan\left(-\frac{\pi}{3}\right)=-\sqrt{3}$.

21. $\sec\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)$

Find the angle θ, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, whose tangent is $\frac{\sqrt{3}}{3}$

$$\tan\theta=\frac{\sqrt{3}}{3}, \quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$$
$$\theta=\frac{\pi}{6}$$

So, $\tan^{-1}\frac{\sqrt{3}}{3}=\frac{\pi}{6}$.

Thus, $\sec\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{2\sqrt{3}}{3}$.

22. $\sin\left(\cot^{-1}\frac{3}{4}\right)$

Since $\cot\theta=\frac{3}{4}$, $0<\theta<\pi$, θ is in quadrant I.

Let $x=3$ and $y=4$. Solve for r: $9+16=r^2$

$$r^2=25$$
$$r=5$$

Thus, $\sin\left(\tan^{-1}\frac{3}{4}\right)=\sin\theta=\frac{y}{r}=\frac{4}{5}$.

23. $\tan\left[\sin^{-1}\left(-\frac{4}{5}\right)\right]$

Since $\sin\theta=-\frac{4}{5}$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, let $y=-4$ and $r=5$. Solve for x: $x^2+16=25$

$$x^2=9$$
$$x=\pm 3$$

Since θ is in quadrant IV, $x=3$.

Thus, $\tan\left[\sin^{-1}\left(-\frac{4}{5}\right)\right]=\tan\theta=\frac{y}{x}=\frac{-4}{3}=-\frac{4}{3}$

24. $f(x) = 2\sin(3x)$

$y = 2\sin(3x)$

$x = 2\sin(3y)$

$\frac{x}{2} = \sin(3y)$

$3y = \sin^{-1}\left(\frac{x}{2}\right)$

$y = \frac{1}{3}\sin^{-1}\left(\frac{x}{2}\right) = f^{-1}(x)$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\frac{\pi}{6} \le x \le \frac{\pi}{6}$, or $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse sine function is $\frac{x}{2}$ and that it must lie in the interval $[-1,1]$. That is,

$-1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$

The domain of $f^{-1}(x)$ is $\{x \mid -2 \le x \le 2\}$, or $[-2,2]$ in interval notation. Recall that the domain of a function is the range of its inverse and the domain of the inverse is the range of the function. Therefore, the range of $f(x)$ is $[-2,2]$.

25. $f(x) = -\cos x + 3$

$y = -\cos x + 3$

$x = -\cos y + 3$

$x - 3 = -\cos y$

$3 - x = \cos y$

$y = \cos^{-1}(3-x) = f^{-1}(x)$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $0 \le x \le \pi$, or $[0,\pi]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse cosine function is $3-x$ and that it must lie in the interval $[-1,1]$. That is,

$-1 \le 3 - x \le 1$

$-4 \le -x \le -2$

$4 \ge x \ge 2$

$2 \le x \le 4$

The domain of $f^{-1}(x)$ is $\{x \mid 2 \le x \le 4\}$, or $[2,4]$ in interval notation. Recall that the domain of a function is the range of its inverse and the domain of the inverse is the range of the function. Therefore, the range of $f(x)$ is $[2,4]$.

26. Let $\theta = \sin^{-1} u$ so that $\sin\theta = u$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $-1 \le u \le 1$. Then,

$$\cos\left(\sin^{-1} u\right) = \cos\theta = \sqrt{\cos^2\theta} = \sqrt{1-\sin^2\theta} = \sqrt{1-u^2}$$

27. Let $\theta = \csc^{-1} u$ so that $\csc\theta = u$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\theta \ne 0$, $|u| \ge 1$. Then,

$$\begin{aligned}\tan\left(\csc^{-1} u\right) &= \tan\theta = \sqrt{\tan^2\theta} \\ &= \sqrt{\frac{1}{\csc^2\theta\cos^2\theta}} \\ &= \sqrt{\frac{1}{u^2}}\sqrt{\frac{1}{1-\sin^2\theta}} \\ &= \sqrt{\frac{1}{u^2}}\sqrt{\frac{1}{1-\frac{1}{\csc^2\theta}}} = \sqrt{\frac{1}{u^2}}\sqrt{\frac{1}{\frac{\csc^2\theta-1}{\csc^2\theta}}} \\ &= \sqrt{\frac{1}{u^2}}\sqrt{\frac{\csc^2\theta}{\csc^2\theta-1}} = \\ &= \frac{|u|}{u\sqrt{u^2-1}}\end{aligned}$$

28. $$\begin{aligned}\tan\theta\cot\theta - \sin^2\theta &= \tan\theta\cdot\frac{1}{\tan\theta} - \sin^2\theta \\ &= 1 - \sin^2\theta = \cos^2\theta\end{aligned}$$

29. $$\begin{aligned}\sin^2\theta(1+\cot^2\theta) &= \sin^2\theta\cdot\csc^2\theta \\ &= \sin^2\theta\cdot\frac{1}{\sin^2\theta} \\ &= 1\end{aligned}$$

30. $$\begin{aligned}5\cos^2\theta+3\sin^2\theta &= 2\cos^2\theta+3\cos^2\theta+3\sin^2\theta\\ &= 2\cos^2\theta+3\left(\cos^2\theta+\sin^2\theta\right)\\ &= 2\cos^2\theta+3\cdot 1\\ &= 3+2\cos^2\theta\end{aligned}$$

31. $$\begin{aligned}\frac{1-\cos\theta}{\sin\theta}+\frac{\sin\theta}{1-\cos\theta} &= \frac{(1-\cos\theta)^2+\sin^2\theta}{\sin\theta(1-\cos\theta)}\\ &= \frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{\sin\theta(1-\cos\theta)}\\ &= \frac{1-2\cos\theta+1}{\sin\theta(1-\cos\theta)}\\ &= \frac{2-2\cos\theta}{\sin\theta(1-\cos\theta)}\\ &= \frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)}\\ &= \frac{2}{\sin\theta}=2\csc\theta\end{aligned}$$

32. $$\begin{aligned}\frac{\cos\theta}{\cos\theta-\sin\theta} &= \frac{\cos\theta}{\cos\theta-\sin\theta}\cdot\frac{\frac{1}{\cos\theta}}{\frac{1}{\cos\theta}}\\ &= \frac{1}{1-\frac{\sin\theta}{\cos\theta}}=\frac{1}{1-\tan\theta}\end{aligned}$$

33. $$\begin{aligned}\frac{\csc\theta}{1+\csc\theta} &= \frac{\frac{1}{\sin\theta}}{1+\frac{1}{\sin\theta}}\cdot\frac{\sin\theta}{\sin\theta}\\ &= \frac{1}{\sin\theta+1}\\ &= \frac{1}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}\\ &= \frac{1-\sin\theta}{1-\sin^2\theta}=\frac{1-\sin\theta}{\cos^2\theta}\end{aligned}$$

34. $$\begin{aligned}\csc\theta-\sin\theta &= \frac{1}{\sin\theta}-\sin\theta\\ &= \frac{1-\sin^2\theta}{\sin\theta}\\ &= \frac{\cos^2\theta}{\sin\theta}\\ &= \cos\theta\cdot\frac{\cos\theta}{\sin\theta}\\ &= \cos\theta\cot\theta\end{aligned}$$

35. $$\begin{aligned}\frac{1-\sin\theta}{\sec\theta} &= \cos\theta(1-\sin\theta)\\ &= \cos\theta(1-\sin\theta)\cdot\frac{1+\sin\theta}{1+\sin\theta}\\ &= \frac{\cos\theta\left(1-\sin^2\theta\right)}{1+\sin\theta}\\ &= \frac{\cos\theta\left(\cos^2\theta\right)}{1+\sin\theta}\\ &= \frac{\cos^3\theta}{1+\sin\theta}\end{aligned}$$

36. $$\begin{aligned}\cot\theta-\tan\theta &= \frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}\\ &= \frac{\cos^2\theta-\sin^2\theta}{\sin\theta\cos\theta}\\ &= \frac{1-\sin^2\theta-\sin^2\theta}{\sin\theta\cos\theta}\\ &= \frac{1-2\sin^2\theta}{\sin\theta\cos\theta}\end{aligned}$$

37. $$\begin{aligned}\frac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta} &= \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\sin\beta}\\ &= \frac{\cos\alpha\cos\beta}{\cos\alpha\sin\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\sin\beta}\\ &= \frac{\cos\beta}{\sin\beta}-\frac{\sin\alpha}{\cos\alpha}\\ &= \cot\beta-\tan\alpha\end{aligned}$$

38. $$\begin{aligned}\frac{\cos(\alpha-\beta)}{\cos\alpha\cos\beta} &= \frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\\ &= \frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\\ &= 1+\tan\alpha\tan\beta\end{aligned}$$

39. $$(1+\cos\theta)\tan\frac{\theta}{2}=(1+\cos\theta)\cdot\frac{\sin\theta}{1+\cos\theta}=\sin\theta$$

40. $$\begin{aligned}2\cot\theta\cot(2\theta) &= 2\cdot\frac{\cos\theta}{\sin\theta}\cdot\frac{\cos(2\theta)}{\sin(2\theta)}\\ &= \frac{2\cos\theta\left(\cos^2\theta-\sin^2\theta\right)}{\sin\theta\left(2\sin\theta\cos\theta\right)}\\ &= \frac{\cos^2\theta-\sin^2\theta}{\sin^2\theta}\\ &= \frac{\cos^2\theta}{\sin^2\theta}-\frac{\sin^2\theta}{\sin^2\theta}\\ &= \cot^2\theta-1\end{aligned}$$

41. $1-8\sin^2\theta\cos^2\theta = 1-2\left(2\sin\theta\cos\theta\right)^2$
$= 1-2\sin^2(2\theta)$
$= \cos(2\cdot 2\theta)$
$= \cos(4\theta)$

42. $\dfrac{\sin(3\theta)\cos\theta-\sin\theta\cos(3\theta)}{\sin(2\theta)} = \dfrac{\sin(3\theta-\theta)}{\sin(2\theta)}$
$= \dfrac{\sin(2\theta)}{\sin(2\theta)}$
$= 1$

43. $\dfrac{\sin(2\theta)+\sin(4\theta)}{\cos(2\theta)+\cos(4\theta)} = \dfrac{2\sin\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta-4\theta}{2}\right)}{2\cos\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta+4\theta}{2}\right)}$
$= \dfrac{2\sin(3\theta)\cos(-\theta)}{2\cos(3\theta)\cos(-\theta)}$
$= \dfrac{\sin(3\theta)}{\cos(3\theta)}$
$= \tan(3\theta)$

44. $\dfrac{\cos(2\theta)-\cos(4\theta)}{\cos(2\theta)+\cos(4\theta)} - \tan\theta\tan(3\theta)$
$= \dfrac{-2\sin(3\theta)\sin(-\theta)}{2\cos(3\theta)\cos(-\theta)} - \tan\theta\tan(3\theta)$
$= \dfrac{2\sin(3\theta)\sin\theta}{2\cos(3\theta)\cos\theta} - \tan\theta\tan(3\theta)$
$= \tan(3\theta)\tan\theta - \tan\theta\tan(3\theta)$
$= 0$

45. $\sin 165^\circ = \sin\left(120^\circ + 45^\circ\right)$
$= \sin 120^\circ\cdot\cos 45^\circ + \cos 120^\circ\cdot\sin 45^\circ$
$= \left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\right)$
$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$
$= \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$

46. $\tan 105^\circ = \tan\left(60^\circ + 45^\circ\right)$
$= \dfrac{\tan 60^\circ + \tan 45^\circ}{1-\tan 60^\circ\tan 45^\circ}$
$= \dfrac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1}$
$= \dfrac{\sqrt{3}+1}{1-\sqrt{3}}\cdot\dfrac{1+\sqrt{3}}{1+\sqrt{3}}$
$= \dfrac{1+2\sqrt{3}+3}{1-3}$
$= \dfrac{4+2\sqrt{3}}{-2}$
$= -2-\sqrt{3}$

47. $\cos\frac{5\pi}{12} = \cos\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right)$
$= \cos\frac{\pi}{4}\cdot\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$
$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$
$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$
$= \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$

48. $\sin\left(-\frac{\pi}{12}\right) = \sin\left(\frac{2\pi}{12}-\frac{3\pi}{12}\right)$
$= \sin\frac{\pi}{6}\cdot\cos\frac{\pi}{4} - \cos\frac{\pi}{6}\cdot\sin\frac{\pi}{4}$
$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$
$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$
$= \frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$

49. $\cos 80^\circ\cdot\cos 20^\circ + \sin 80^\circ\cdot\sin 20^\circ = \cos\left(80^\circ - 20^\circ\right)$
$= \cos 60^\circ$
$= \frac{1}{2}$

50. $\sin 70^\circ\cdot\cos 40^\circ - \cos 70^\circ\cdot\sin 40^\circ = \sin\left(70^\circ - 40^\circ\right)$
$= \sin 30^\circ$
$= \frac{1}{2}$

51. $\tan\frac{\pi}{8}=\tan\left(\frac{\frac{\pi}{4}}{2}\right)=\sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}}=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}}$

$$=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$
$$=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}}}$$
$$=\sqrt{\frac{\left(2-\sqrt{2}\right)^2}{4}}$$
$$=\frac{2-\sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{2\sqrt{2}-2}{2}$$
$$=\sqrt{2}-1$$

52. $\sin\frac{5\pi}{8}=\sin\left(\frac{\frac{5\pi}{4}}{2}\right)=\sqrt{\frac{1-\cos\frac{5\pi}{4}}{2}}=\sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}}$

$$=\sqrt{\frac{2+\sqrt{2}}{4}}$$
$$=\frac{\sqrt{2+\sqrt{2}}}{2}$$

53. $\sin\alpha=\frac{4}{5},\ 0<\alpha<\frac{\pi}{2};\quad \sin\beta=\frac{5}{13},\ \frac{\pi}{2}<\beta<\pi$

$\cos\alpha=\frac{3}{5},\ \tan\alpha=\frac{4}{3},\ \cos\beta=-\frac{12}{13},\ \tan\beta=-\frac{5}{12},$

$0<\frac{\alpha}{2}<\frac{\pi}{4},\quad \frac{\pi}{4}<\frac{\beta}{2}<\frac{\pi}{2}$

a. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$$=\left(\frac{4}{5}\right)\cdot\left(-\frac{12}{13}\right)+\left(\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right)$$
$$=\frac{-48+15}{65}=-\frac{33}{65}$$

b. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$$=\left(\frac{3}{5}\right)\cdot\left(-\frac{12}{13}\right)-\left(\frac{4}{5}\right)\cdot\left(\frac{5}{13}\right)$$
$$=\frac{-36-20}{65}=-\frac{56}{65}$$

c. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$$=\left(\frac{4}{5}\right)\cdot\left(-\frac{12}{13}\right)-\left(\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right)$$
$$=\frac{-48-15}{65}=-\frac{63}{65}$$

d. $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$$=\frac{\frac{4}{3}+\left(-\frac{5}{12}\right)}{1-\left(\frac{4}{3}\right)\cdot\left(-\frac{5}{12}\right)}$$
$$=\frac{\frac{11}{12}}{\frac{14}{9}}=\frac{11}{12}\cdot\frac{9}{14}=\frac{33}{56}$$

e. $\sin(2\alpha)=2\sin\alpha\cos\alpha=2\cdot\frac{4}{5}\cdot\frac{3}{5}=\frac{24}{25}$

f. $\cos(2\beta)=\cos^2\beta-\sin^2\beta$
$$=\left(-\frac{12}{13}\right)^2-\left(\frac{5}{13}\right)^2=\frac{144}{169}-\frac{25}{169}=\frac{119}{169}$$

g. $\sin\frac{\beta}{2}=\sqrt{\frac{1-\cos\beta}{2}}$
$$=\sqrt{\frac{1-\left(-\frac{12}{13}\right)}{2}}$$
$$=\sqrt{\frac{\frac{25}{13}}{2}}=\sqrt{\frac{25}{26}}=\frac{5}{\sqrt{26}}=\frac{5\sqrt{26}}{26}$$

h. $\cos\frac{\alpha}{2}=\sqrt{\frac{1+\cos\alpha}{2}}$
$$=\sqrt{\frac{1+\frac{3}{5}}{2}}=\sqrt{\frac{\frac{8}{5}}{2}}=\sqrt{\frac{4}{5}}=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$$

54. $\sin\alpha = -\frac{3}{5}, \pi < \alpha < \frac{3\pi}{2}; \cos\beta = \frac{12}{13}, \frac{3\pi}{2} < \beta < 2\pi$

$\cos\alpha = -\frac{4}{5}, \tan\alpha = \frac{3}{4}, \sin\beta = -\frac{5}{13}, \tan\beta = -\frac{5}{12},$

$\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}, \quad \frac{3\pi}{4} < \frac{\beta}{2} < \pi$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$$= \left(-\frac{3}{5}\right)\cdot\left(\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\cdot\left(-\frac{5}{13}\right) = \frac{-36+20}{65} = -\frac{16}{65}$$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$$= \left(-\frac{4}{5}\right)\cdot\left(\frac{12}{13}\right) - \left(-\frac{3}{5}\right)\cdot\left(-\frac{5}{13}\right) = \frac{-48-15}{65} = -\frac{63}{65}$$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$$= \left(-\frac{3}{5}\right)\cdot\left(\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\cdot\left(-\frac{5}{13}\right) = \frac{-36-20}{65} = -\frac{56}{65}$$

d. $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$
$$= \frac{\frac{3}{4} + \left(-\frac{5}{12}\right)}{1 - \frac{3}{4}\left(-\frac{5}{12}\right)} = \frac{\frac{1}{3}}{\frac{21}{16}} = \frac{1}{3}\cdot\frac{16}{21} = \frac{16}{63}$$

e. $\sin(2\alpha) = 2\sin\alpha\cos\alpha$
$$= 2\cdot\left(-\frac{3}{5}\right)\cdot\left(-\frac{4}{5}\right) = \frac{24}{25}$$

f. $\cos(2\beta) = \cos^2\beta - \sin^2\beta$
$$= \left(\frac{12}{13}\right)^2 - \left(-\frac{5}{13}\right)^2 = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$$

g. $\sin\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{2}}$
$$= \sqrt{\frac{1-\frac{12}{13}}{2}} = \sqrt{\frac{\frac{1}{13}}{2}} = \sqrt{\frac{1}{26}} = \frac{1}{\sqrt{26}} = \frac{\sqrt{26}}{26}$$

h. $\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$
$$= -\sqrt{\frac{1+\left(-\frac{4}{5}\right)}{2}} = -\sqrt{\frac{\frac{1}{5}}{2}} = -\sqrt{\frac{1}{10}} = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$$

55. $\tan\alpha = \frac{3}{4}, \pi < \alpha < \frac{3\pi}{2}; \ \tan\beta = \frac{12}{5}, 0 < \beta < \frac{\pi}{2}$

$\sin\alpha = -\frac{3}{5}, \cos\alpha = -\frac{4}{5}, \sin\beta = \frac{12}{13}, \cos\beta = \frac{5}{13},$

$\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}, \quad 0 < \frac{\beta}{2} < \frac{\pi}{4}$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$$= \left(-\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\cdot\left(\frac{12}{13}\right) = -\frac{15}{65} - \frac{48}{65} = -\frac{63}{65}$$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$$= \left(-\frac{4}{5}\right)\cdot\left(\frac{5}{13}\right) - \left(-\frac{3}{5}\right)\cdot\left(\frac{12}{13}\right) = -\frac{20}{65} + \frac{36}{65} = \frac{16}{65}$$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$$= \left(-\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right) - \left(-\frac{4}{5}\right)\cdot\left(\frac{12}{13}\right) = -\frac{15}{65} + \frac{48}{65} = \frac{33}{65}$$

d. $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$=\dfrac{\frac{3}{4}+\frac{12}{5}}{1-\left(\frac{3}{4}\right)\left(\frac{12}{5}\right)}$
$=\dfrac{\frac{63}{20}}{-\frac{4}{5}}=\dfrac{63}{20}\left(-\dfrac{5}{4}\right)=-\dfrac{63}{16}$

e. $\sin(2\alpha)=2\sin\alpha\cos\alpha=2\left(-\dfrac{3}{5}\right)\left(-\dfrac{4}{5}\right)=\dfrac{24}{25}$

f. $\cos(2\beta)=\cos^2\beta-\sin^2\beta$
$=\left(\dfrac{5}{13}\right)^2-\left(\dfrac{12}{13}\right)^2=\dfrac{25}{169}-\dfrac{144}{169}=-\dfrac{119}{169}$

g. $\sin\dfrac{\beta}{2}=\sqrt{\dfrac{1-\cos\beta}{2}}$
$=\sqrt{\dfrac{1-\frac{5}{13}}{2}}=\sqrt{\dfrac{\frac{8}{13}}{2}}=\sqrt{\dfrac{4}{13}}=\dfrac{2}{\sqrt{13}}=\dfrac{2\sqrt{13}}{13}$

h. $\cos\dfrac{\alpha}{2}=-\sqrt{\dfrac{1+\cos\alpha}{2}}$
$=-\sqrt{\dfrac{1+\left(-\frac{4}{5}\right)}{2}}$
$=-\sqrt{\dfrac{\frac{1}{5}}{2}}=-\sqrt{\dfrac{1}{10}}=-\dfrac{1}{\sqrt{10}}=-\dfrac{\sqrt{10}}{10}$

56. $\sec\alpha=2,-\dfrac{\pi}{2}<\alpha<0;\ \sec\beta=3,\dfrac{3\pi}{2}<\beta<2\pi$

$\sin\alpha=-\dfrac{\sqrt{3}}{2},\ \cos\alpha=\dfrac{1}{2},\ \tan\alpha=-\sqrt{3},$

$\sin\beta=-\dfrac{2\sqrt{2}}{3},\ \cos\beta=\dfrac{1}{3},\ \tan\beta=-2\sqrt{2},$

$-\dfrac{\pi}{4}<\dfrac{\alpha}{2}<0,\quad \dfrac{3\pi}{4}<\dfrac{\beta}{2}<\pi$

a. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$=-\dfrac{\sqrt{3}}{2}\left(\dfrac{1}{3}\right)+\dfrac{1}{2}\left(-\dfrac{2\sqrt{2}}{3}\right)$
$=\dfrac{-\sqrt{3}-2\sqrt{2}}{6}$

b. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$=\dfrac{1}{2}\cdot\dfrac{1}{3}-\left(-\dfrac{\sqrt{3}}{2}\right)\left(-\dfrac{2\sqrt{2}}{3}\right)$
$=\dfrac{1-2\sqrt{6}}{6}$

c. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$=-\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{3}-\dfrac{1}{2}\cdot\left(-\dfrac{2\sqrt{2}}{3}\right)$
$=\dfrac{-\sqrt{3}+2\sqrt{2}}{6}$

d. $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$=\dfrac{-\sqrt{3}+\left(-2\sqrt{2}\right)}{1-\left(-\sqrt{3}\right)\left(-2\sqrt{2}\right)}$
$=\left(\dfrac{-\sqrt{3}-2\sqrt{2}}{1-2\sqrt{6}}\right)\cdot\left(\dfrac{1+2\sqrt{6}}{1+2\sqrt{6}}\right)$
$=\dfrac{-9\sqrt{3}-8\sqrt{2}}{-23}$
$=\dfrac{8\sqrt{2}+9\sqrt{3}}{23}$

e. $\sin(2\alpha)=2\sin\alpha\cos\alpha=2\left(-\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1}{2}\right)=-\dfrac{\sqrt{3}}{2}$

f. $\cos(2\beta)=\cos^2\beta-\sin^2\beta$
$=\left(\dfrac{1}{3}\right)^2-\left(-\dfrac{2\sqrt{2}}{3}\right)^2=\dfrac{1}{9}-\dfrac{8}{9}=-\dfrac{7}{9}$

g. $\sin\dfrac{\beta}{2}=\sqrt{\dfrac{1-\cos\beta}{2}}$
$=\sqrt{\dfrac{1-\frac{1}{3}}{2}}=\sqrt{\dfrac{\frac{2}{3}}{2}}=\sqrt{\dfrac{1}{3}}=\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}$

h. $\cos\dfrac{\alpha}{2}=\sqrt{\dfrac{1+\cos\alpha}{2}}$
$=\sqrt{\dfrac{1+\frac{1}{2}}{2}}=\sqrt{\dfrac{\frac{3}{2}}{2}}=\sqrt{\dfrac{3}{4}}=\dfrac{\sqrt{3}}{2}$

57. $\sin\alpha = -\frac{2}{3}, \pi < \alpha < \frac{3\pi}{2}; \cos\beta = -\frac{2}{3}, \pi < \beta < \frac{3\pi}{2}$

$\cos\alpha = -\frac{\sqrt{5}}{3}, \tan\alpha = \frac{2\sqrt{5}}{5}, \sin\beta = -\frac{\sqrt{5}}{3},$

$\tan\beta = \frac{\sqrt{5}}{2}, \frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}, \frac{\pi}{2} < \frac{\beta}{2} < \frac{3\pi}{4}$

a. $$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) + \left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) \\ &= \frac{4}{9} + \frac{5}{9} \\ &= 1\end{aligned}$$

b. $$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) - \left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) \\ &= \frac{2\sqrt{5}}{9} - \frac{2\sqrt{5}}{9} \\ &= 0\end{aligned}$$

c. $$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ &= \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) - \left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) \\ &= \frac{4}{9} - \frac{5}{9} \\ &= -\frac{1}{9}\end{aligned}$$

d. $$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ &= \frac{\frac{2\sqrt{5}}{5} + \frac{\sqrt{5}}{2}}{1 - \frac{2\sqrt{5}}{5}\cdot\frac{\sqrt{5}}{2}} \\ &= \frac{\frac{4\sqrt{5}+5\sqrt{5}}{10}}{1-1} \\ &= \frac{\frac{9\sqrt{5}}{10}}{0}; \quad \text{Undefined}\end{aligned}$$

e. $$\begin{aligned}\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\ &= 2\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) = \frac{4\sqrt{5}}{9}\end{aligned}$$

f. $$\begin{aligned}\cos(2\beta) &= \cos^2\beta - \sin^2\beta \\ &= \left(-\frac{2}{3}\right)^2 - \left(-\frac{\sqrt{5}}{3}\right)^2 = \frac{4}{9} - \frac{5}{9} = -\frac{1}{9}\end{aligned}$$

g. $$\begin{aligned}\sin\frac{\beta}{2} &= \sqrt{\frac{1-\cos\beta}{2}} \\ &= \sqrt{\frac{1-\left(-\frac{2}{3}\right)}{2}} = \sqrt{\frac{\frac{5}{3}}{2}} = \sqrt{\frac{5}{6}} = \frac{\sqrt{30}}{6}\end{aligned}$$

h. $$\begin{aligned}\cos\frac{\alpha}{2} &= -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\left(-\frac{\sqrt{5}}{3}\right)}{2}} \\ &= -\sqrt{\frac{\frac{3-\sqrt{5}}{3}}{2}} \\ &= -\sqrt{\frac{3-\sqrt{5}}{6}} \\ &= -\frac{\sqrt{6\left(3-\sqrt{5}\right)}}{6} \\ &= -\frac{\sqrt{6}\sqrt{3-\sqrt{5}}}{6}\end{aligned}$$

58. $\cos\left(\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{1}{2}\right)$

Let $\alpha = \sin^{-1}\frac{3}{5}$ and $\beta = \cos^{-1}\frac{1}{2}$. α is in quadrant I; β is in quadrant I. Then $\sin\alpha = \frac{3}{5}$, $0 \le \alpha \le \frac{\pi}{2}$, and $\cos\beta = \frac{1}{2}$, $0 \le \beta \le \frac{\pi}{2}$.

$$\begin{aligned}\cos\alpha &= \sqrt{1-\sin^2\alpha} \\ &= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}\end{aligned}$$

$$\begin{aligned}\sin\beta &= \sqrt{1-\cos^2\beta} \\ &= \sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\cos\left(\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{1}{2}\right) &= \cos(\alpha-\beta) \\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \frac{4}{5}\cdot\frac{1}{2} + \frac{3}{5}\cdot\frac{\sqrt{3}}{2} \\ &= \frac{4}{10} + \frac{3\sqrt{3}}{10} = \frac{4+3\sqrt{3}}{10}\end{aligned}$$

59. $\sin\left(\cos^{-1}\frac{5}{13}-\cos^{-1}\frac{4}{5}\right)$

Let $\alpha=\cos^{-1}\frac{5}{13}$ and $\beta=\cos^{-1}\frac{4}{5}$. α is in quadrant I; β is in quadrant I. Then $\cos\alpha=\frac{5}{13}$, $0\le\alpha\le\frac{\pi}{2}$, and $\cos\beta=\frac{4}{5}$, $0\le\beta\le\frac{\pi}{2}$.

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{169}}=\sqrt{\frac{144}{169}}=\frac{12}{13}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\begin{aligned}\sin\left(\cos^{-1}\frac{5}{13}-\cos^{-1}\frac{4}{5}\right)&=\sin(\alpha-\beta)\\&=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\&=\frac{12}{13}\cdot\frac{4}{5}-\frac{5}{13}\cdot\frac{3}{5}\\&=\frac{48}{65}-\frac{15}{65}=\frac{33}{65}\end{aligned}$$

60. $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)-\tan^{-1}\frac{3}{4}\right]$

Let $\alpha=\sin^{-1}\left(-\frac{1}{2}\right)$ and $\beta=\tan^{-1}\frac{3}{4}$. α is in quadrant IV; β is in quadrant I. Then, $\sin\alpha=-\frac{1}{2}$, $0\le\alpha\le\frac{\pi}{2}$, and $\tan\beta=\frac{3}{4}$, $0<\beta<\frac{\pi}{2}$.

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(-\frac{1}{2}\right)^2}=\sqrt{1-\frac{1}{4}}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

$$\tan\alpha=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$$

$$\begin{aligned}\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right]&=\tan(\alpha-\beta)\\&=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\\&=\frac{-\frac{\sqrt{3}}{3}-\frac{3}{4}}{1+\left(-\frac{\sqrt{3}}{3}\right)\left(\frac{3}{4}\right)}\\&=\frac{\frac{-4\sqrt{3}-9}{12}}{1-\frac{3\sqrt{3}}{12}}\\&=\frac{-9-4\sqrt{3}}{12-3\sqrt{3}}\cdot\frac{12+3\sqrt{3}}{12+3\sqrt{3}}\\&=\frac{-144-75\sqrt{3}}{117}\\&=\frac{-48-25\sqrt{3}}{39}\\&=-\frac{48+25\sqrt{3}}{39}\end{aligned}$$

61. $\cos\left[\tan^{-1}(-1)+\cos^{-1}\left(-\frac{4}{5}\right)\right]$

Let $\alpha=\tan^{-1}(-1)$ and $\beta=\cos^{-1}\left(-\frac{4}{5}\right)$. α is in quadrant IV; β is in quadrant II. Then $\tan\alpha=-1$, $-\frac{\pi}{2}<\alpha<0$, and $\cos\beta=-\frac{4}{5}$, $\frac{\pi}{2}\le\beta\le\pi$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}=\sqrt{1+(-1)^2}=\sqrt{2}$$

$$\cos\alpha=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

$$\sin\alpha=-\sqrt{1-\cos^2\alpha}=-\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}=-\sqrt{1-\frac{1}{2}}=-\sqrt{\frac{1}{2}}=-\frac{\sqrt{2}}{2}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(-\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\cos\left[\tan^{-1}(-1)+\cos^{-1}\left(-\frac{4}{5}\right)\right]=\cos(\alpha+\beta)$$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{4}{5}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{3}{5}\right)$$
$$=\frac{-4\sqrt{2}}{10}+\frac{3\sqrt{2}}{10}$$
$$=-\frac{\sqrt{2}}{10}$$

62. $\sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha=\cos^{-1}\left(-\frac{3}{5}\right)$. α is in quadrant II. Then

$\cos\alpha=-\frac{3}{5}$, $\frac{\pi}{2}\le\alpha\le\pi$.

$$\sin\alpha=\sqrt{1-\cos^2\alpha}$$
$$=\sqrt{1-\left(-\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$
$$\sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]=\sin 2\alpha$$
$$=2\sin\alpha\cos\alpha$$
$$=2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)=-\frac{24}{25}$$

63. $\cos\left(2\tan^{-1}\frac{4}{3}\right)$

Let $\alpha=\tan^{-1}\frac{4}{3}$. α is in quadrant I. Then

$\tan\alpha=\frac{4}{3}$, $0<\alpha<\frac{\pi}{2}$.

$$\sec\alpha=\sqrt{\tan^2\alpha+1}$$
$$=\sqrt{\left(\frac{4}{3}\right)^2+1}=\sqrt{\frac{16}{9}+1}=\sqrt{\frac{25}{9}}=\frac{5}{3}$$
$$\cos\alpha=\frac{3}{5}$$
$$\cos\left(2\tan^{-1}\frac{4}{3}\right)=\cos(2\alpha)$$
$$=2\cos^2\alpha-1$$
$$=2\left(\frac{3}{5}\right)^2-1=2\left(\frac{9}{25}\right)-1=-\frac{7}{25}$$

64. $\cos\theta=\frac{1}{2}$

$\theta=\frac{\pi}{3}+2k\pi$ or $\theta=\frac{5\pi}{3}+2k\pi$, k is any integer

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{\pi}{3},\ \frac{5\pi}{3}\right\}$.

65. $\tan\theta+\sqrt{3}=0$

$\tan\theta=-\sqrt{3}$

$\theta=\frac{2\pi}{3}+k\pi$, k is any integer

On the interval $0\le\theta<2\pi$, the solution set is $\left\{\frac{2\pi}{3},\ \frac{5\pi}{3}\right\}$.

66. $\sin(2\theta)+1=0$

$\sin(2\theta)=-1$

$2\theta=\frac{3\pi}{2}+2k\pi$

$\theta=\frac{3\pi}{4}+k\pi$, k is any integer

On the interval $0\le\theta<2\pi$, the solution set is $\left\{\frac{3\pi}{4},\ \frac{7\pi}{4}\right\}$

67. $\tan(2\theta)=0$

$2\theta=0+k\pi$

$\theta=\frac{k\pi}{2}$, where k is any integer

On the interval $0\le\theta<2\pi$, the solution set is $\left\{0,\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\}$.

68. $\sec^2\theta=4$

$\sec\theta=\pm 2$

$\cos\theta=\pm\frac{1}{2}$

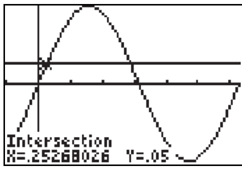
$\theta=\frac{\pi}{3}+k\pi$ or $\theta=\frac{2\pi}{3}+k\pi$,

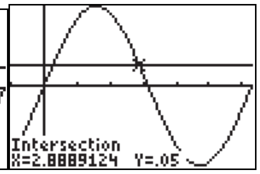
where k is any integer

On the interval $0\le\theta<2\pi$, the solution set is $\left\{\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}\right\}$.

69. $0.2\sin\theta = 0.05$

Find the intersection of $Y_1 = 0.2\sin\theta$ and $Y_2 = 0.05$:





On the interval $0 \le \theta < 2\pi$, $x \approx 0.25$ or $x \approx 2.89$

The solution set is $\{0.25, 2.89\}$.

70.
$$\sin\theta + \sin(2\theta) = 0$$
$$\sin\theta + 2\sin\theta\cos\theta = 0$$
$$\sin\theta(1 + 2\cos\theta) = 0$$
$1 + 2\cos\theta = 0$ or $\sin\theta = 0$

$\cos\theta = -\frac{1}{2}$ $\quad\theta = 0, \pi$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

On $0 \le \theta < 2\pi$, the solution set is $\left\{0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}\right\}$.

71.
$$\sin(2\theta) - \cos\theta - 2\sin\theta + 1 = 0$$
$$2\sin\theta\cos\theta - \cos\theta - 2\sin\theta + 1 = 0$$
$$\cos\theta(2\sin\theta - 1) - 1(2\sin\theta - 1) = 0$$
$$(2\sin\theta - 1)(\cos\theta - 1) = 0$$
$\sin\theta = \frac{1}{2}$ or $\cos\theta = 1$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad\theta = 0$

On $0 \le \theta < 2\pi$, the solution set is $\left\{0, \frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

72.
$$2\sin^2\theta - 3\sin\theta + 1 = 0$$
$$(2\sin\theta - 1)(\sin\theta - 1) = 0$$
$2\sin\theta - 1 = 0$ or $\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{2}$ $\quad\sin\theta = 1$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad\theta = \frac{\pi}{2}$

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$.

73.
$$4\sin^2\theta = 1 + 4\cos\theta$$
$$4\left(1 - \cos^2\theta\right) = 1 + 4\cos\theta$$
$$4 - 4\cos^2\theta = 1 + 4\cos\theta$$
$$4\cos^2\theta + 4\cos\theta - 3 = 0$$
$$(2\cos\theta - 1)(2\cos\theta + 3) = 0$$
$2\cos\theta - 1 = 0$ or $2\cos\theta + 3 = 0$

$\cos\theta = \frac{1}{2}$ $\quad\cos\theta = -\frac{3}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ $\quad$(not possible)

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

74.
$$\sin(2\theta) = \sqrt{2}\cos\theta$$
$$2\sin\theta\cos\theta = \sqrt{2}\cos\theta$$
$$2\sin\theta\cos\theta - \sqrt{2}\cos\theta = 0$$
$$\cos\theta\left(2\sin\theta - \sqrt{2}\right) = 0$$
$\cos\theta = 0$ or $2\sin\theta - \sqrt{2} = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad\sin\theta = \frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}\right\}$.

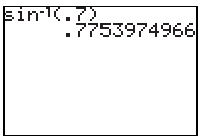
75. $\sin\theta - \cos\theta = 1$

Divide each side by $\sqrt{2}$:
$$\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$$
Rewrite in the difference of two angles form where $\cos\phi = \frac{1}{\sqrt{2}}$, $\sin\phi = \frac{1}{\sqrt{2}}$, and $\phi = \frac{\pi}{4}$:
$$\sin\theta\cos\phi - \cos\theta\sin\phi = \frac{1}{\sqrt{2}}$$
$$\sin(\theta - \phi) = \frac{\sqrt{2}}{2}$$

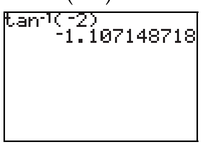
$$\theta - \phi = \frac{\pi}{4} \text{ or } \theta - \phi = \frac{3\pi}{4}$$
$$\theta - \frac{\pi}{4} = \frac{\pi}{4} \text{ or } \theta - \frac{\pi}{4} = \frac{3\pi}{4}$$
$$\theta = \frac{\pi}{2} \text{ or } \theta = \pi$$

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{2}, \pi\right\}$.

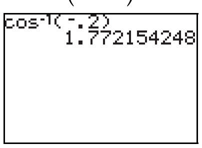
76. $\sin^{-1}(0.7) \approx 0.78$



77. $\tan^{-1}(-2) \approx -1.11$

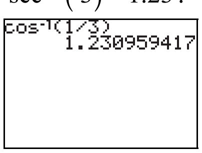


78. $\cos^{-1}(-0.2) \approx 1.77$



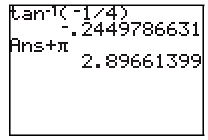
79. $\sec^{-1}(3) = \cos^{-1}\left(\frac{1}{3}\right)$

We seek the angle θ, $0 \le \theta \le \pi$, whose cosine equals $\frac{1}{3}$. Now $\cos\theta = \frac{1}{3}$, so θ lies in quadrant I. The calculator yields $\cos^{-1}\frac{1}{3} \approx 1.23$, which is an angle in quadrant I, so $\sec^{-1}(3) \approx 1.23$.



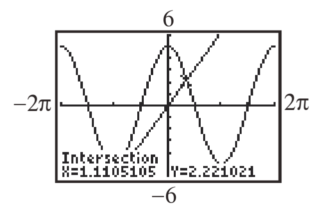
80. $\cot^{-1}(-4) = \tan^{-1}\left(-\frac{1}{4}\right)$

We seek the angle θ, $0 \le \theta \le \pi$, whose tangent equals $-\frac{1}{4}$. Now $\tan\theta = -\frac{1}{4}$, so θ lies in quadrant II. The calculator yields $\tan^{-1}\left(-\frac{1}{4}\right) \approx -0.24$, which is an angle in quadrant IV. Since θ lies in quadrant II, $\theta \approx -0.24 + \pi \approx 2.90$. Therefore, $\cot^{-1}(-4) \approx 2.90$.



81. $2x = 5\cos x$

Find the intersection of $Y_1 = 2x$ and $Y_2 = 5\cos x$:

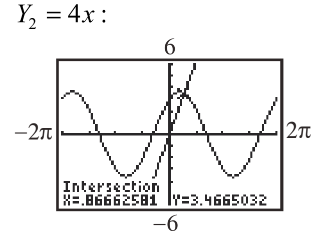


$x \approx 1.11$

The solution set is $\{1.11\}$.

82. $2\sin x + 3\cos x = 4x$

Find the intersection of $Y_1 = 2\sin x + 3\cos x$ and $Y_2 = 4x$:

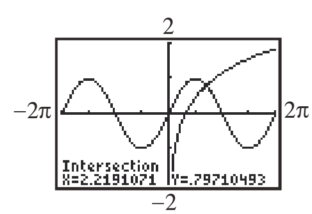


$x \approx 0.87$.

The solution set is $\{0.87\}$.

83. $\sin x = \ln x$

Find the intersection of $Y_1 = \sin x$ and $Y_2 = \ln x$:



$x \approx 2.22$

The solution set is $\{2.22\}$.

84. $-3\sin^{-1} x = \pi$

$$\sin^{-1} x = -\frac{\pi}{3}$$
$$x = \sin\left(-\frac{\pi}{3}\right)$$
$$= -\frac{\sqrt{3}}{2}$$

The solution set is $\left\{-\frac{\sqrt{3}}{2}\right\}$.

85. $2\cos^{-1} x + \pi = 4\cos^{-1} x$

$$-2\cos^{-1} x + \pi = 0$$
$$-2\cos^{-1} x = -\pi$$
$$\cos^{-1} x = \frac{\pi}{2}$$
$$x = \cos\frac{\pi}{2} = 0$$

The solution set is $\{0\}$.

86. Using a half-angle formula:

$$\sin 15° = \sin\left(\frac{30°}{2}\right)$$
$$= \sqrt{\frac{1-\cos 30°}{2}}$$
$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

Note: since 15º lies in quadrant I, we have $\sin 15° > 0$.

Using a difference formula:

$$\sin 15° = \sin(45° - 30°)$$
$$= \sin(45°)\cos(30°) - \cos(45°)\sin(30°)$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$$

Verifying equality:

$$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$
$$= \frac{\sqrt{2}\cdot\sqrt{3}-\sqrt{2}}{4}$$
$$= \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}$$
$$= \sqrt{\left(\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}\right)^2}$$
$$= \sqrt{\frac{2\left(3-2\sqrt{3}+1\right)}{16}}$$
$$= \sqrt{\frac{2\left(4-2\sqrt{3}\right)}{16}}$$
$$= \sqrt{\frac{2\cdot 2\left(2-\sqrt{3}\right)}{16}}$$
$$= \sqrt{\frac{2-\sqrt{3}}{4}}$$
$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

87. Given the value of $\cos\theta$, the most efficient Double-angle Formula to use is $\cos(2\theta) = 2\cos^2\theta - 1$.

Chapter 7 Test

1. Let $\theta = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$. We seek the angle θ, such that $0 \le \theta \le \pi$ and $\theta \ne \frac{\pi}{2}$, whose secant equals $\frac{2}{\sqrt{3}}$. The only value in the restricted range with a secant of $\frac{2}{\sqrt{3}}$ is $\frac{\pi}{6}$. Thus, $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$.

2. Let $\theta = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. We seek the angle θ, such that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{2}}{2}$. The only value in the restricted range with a sine of $-\frac{2}{\sqrt{2}}$ is $-\frac{\pi}{4}$. Thus, $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

3. $\sin^{-1}\left(\sin\frac{11\pi}{5}\right)$ follows the form of the equation $f^{-1}\left(f(x)\right) = \sin^{-1}\left(\sin(x)\right) = x$, but because $\frac{11\pi}{5}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we cannot directly use the equation.
We need to find an angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin\frac{11\pi}{5} = \sin\theta$. The angle $\frac{11\pi}{5}$ is in quadrant I. The reference angle of $\frac{11\pi}{5}$ is $\frac{\pi}{5}$ and $\sin\frac{11\pi}{5} = \sin\frac{\pi}{5}$. Since $\frac{\pi}{5}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can apply the equation above and get $\sin^{-1}\left(\sin\frac{11\pi}{5}\right) = \frac{\pi}{5}$.

4. $\tan\left(\tan^{-1}\frac{7}{3}\right)$ follows the form $f\left(f^{-1}(x)\right) = \tan\left(\tan^{-1}x\right) = x$. Since the domain of the inverse tangent is all real numbers, we can directly apply this equation to get $\tan\left(\tan^{-1}\frac{7}{3}\right) = \frac{7}{3}$.

5. $\cot\left(\csc^{-1}\sqrt{10}\right)$

Since $\csc^{-1}\theta = \frac{r}{y} = \sqrt{10}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, let $r = \sqrt{10}$ and $y = 1$. Solve for *x*:

$$x^2 + 1^2 = \left(\sqrt{10}\right)^2$$
$$x^2 + 1 = 10$$
$$x^2 = 9$$
$$x = 3$$

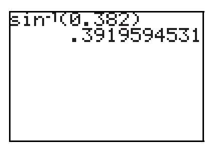
θ is in quadrant I.

Thus, $\cot\left(\csc^{-1}\sqrt{10}\right) = \cot\theta = \frac{x}{y} = \frac{3}{1} = 3$.

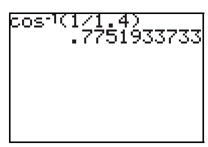
6. Let $\theta = \cos^{-1}\left(-\frac{3}{4}\right)$.

$$\sec\left[\cos^{-1}\left(-\frac{3}{4}\right)\right] = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\cos\left[\cos^{-1}\left(-\frac{3}{4}\right)\right]} = \frac{1}{-\frac{3}{4}} = -\frac{4}{3}$$

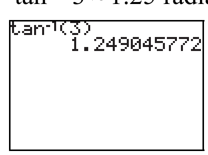
7. $\sin^{-1}(0.382) \approx 0.39$ radian



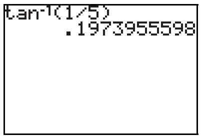
8. $\sec^{-1}1.4 = \cos^{-1}\left(\frac{1}{1.4}\right) \approx 0.78$ radian



9. $\tan^{-1}3 \approx 1.25$ radians



10. $\cot^{-1} 5 = \tan^{-1}\left(\frac{1}{5}\right) \approx 0.20$ radian



11. $$\frac{\csc\theta + \cot\theta}{\sec\theta + \tan\theta}$$
$$= \frac{\csc\theta + \cot\theta}{\sec\theta + \tan\theta} \cdot \frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta}$$
$$= \frac{\csc^2\theta - \cot^2\theta}{(\sec\theta + \tan\theta)(\csc\theta - \cot\theta)}$$
$$= \frac{1}{(\sec\theta + \tan\theta)(\csc\theta - \cot\theta)}$$
$$= \frac{1}{(\sec\theta + \tan\theta)(\csc\theta - \cot\theta)} \cdot \frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}$$
$$= \frac{\sec\theta - \tan\theta}{(\sec^2\theta - \tan^2\theta)(\csc\theta - \cot\theta)}$$
$$= \frac{\sec\theta - \tan\theta}{\csc\theta - \cot\theta}$$

12. $$\sin\theta\tan\theta + \cos\theta = \sin\theta \cdot \frac{\sin\theta}{\cos\theta} + \cos\theta$$
$$= \frac{\sin^2\theta}{\cos\theta} + \frac{\cos^2\theta}{\cos\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta}$$
$$= \frac{1}{\cos\theta}$$
$$= \sec\theta$$

13. $$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{1}{\sin\theta\cos\theta}$$
$$= \frac{2}{2\sin\theta\cos\theta}$$
$$= \frac{2}{\sin(2\theta)}$$
$$= 2\csc(2\theta)$$

14. $$\frac{\sin(\alpha+\beta)}{\tan\alpha + \tan\beta}$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{1} \cdot \frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$$
$$= \cos\alpha\cos\beta$$

15. $$\sin(3\theta)$$
$$= \sin(\theta + 2\theta)$$
$$= \sin\theta\cos(2\theta) + \cos\theta\sin(2\theta)$$
$$= \sin\theta\left(\cos^2\theta - \sin^2\theta\right) + \cos\theta \cdot 2\sin\theta\cos\theta$$
$$= \sin\theta\cos^2\theta - \sin^3\theta + 2\sin\theta\cos^2\theta$$
$$= 3\sin\theta\cos^2\theta - \sin^3\theta$$
$$= 3\sin\theta\left(1 - \sin^2\theta\right) - \sin^3\theta$$
$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

16. $$\frac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} = \frac{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}$$
$$= \frac{\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}$$
$$= \frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta + \cos^2\theta}$$
$$= \frac{-\cos(2\theta)}{1}$$
$$= -\left(2\cos^2\theta - 1\right)$$
$$= 1 - 2\cos^2\theta$$

17. $$\cos 15° = \cos(45° - 30°)$$
$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{2}}{4}\left(\sqrt{3}+1\right)$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4} \quad \text{or} \quad \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

18. $$\tan 75° = \tan(45° + 30°)$$
$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$
$$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1 \cdot \frac{\sqrt{3}}{3}}$$
$$= \frac{3+\sqrt{3}}{3-\sqrt{3}}$$
$$= \frac{3+\sqrt{3}}{3-\sqrt{3}} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}}$$
$$= \frac{9+6\sqrt{3}+3}{3^2-3}$$
$$= \frac{12+6\sqrt{3}}{6}$$
$$= 2+\sqrt{3}$$

19. $$\sin\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right)$$

Let $\theta = \cos^{-1}\frac{3}{5}$. Since $0 < \theta < \frac{\pi}{2}$ (from the range of $\cos^{-1} x$),
$$\sin\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1-\cos\theta}{2}}$$
$$= \sqrt{\frac{1-\cos\left(\cos^{-1}\frac{3}{5}\right)}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}}$$
$$= \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$$

20. $$\tan\left(2\sin^{-1}\frac{6}{11}\right)$$

Let $\theta = \sin^{-1}\frac{6}{11}$. Then $\sin\theta = \frac{6}{11}$ and θ lies in quadrant I. Since $\sin\theta = \frac{y}{r} = \frac{6}{11}$, let $y = 6$ and $r = 11$, and solve for *x*:
$$x^2 + 6^2 = 11^2$$
$$x^2 = 85$$
$$x = \sqrt{85}$$

$$\tan\theta = \frac{y}{x} = \frac{6}{\sqrt{85}} = \frac{6\sqrt{85}}{85}$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{6\sqrt{85}}{85}\right)}{1-\left(\frac{6\sqrt{85}}{85}\right)^2}$$
$$= \frac{\frac{12\sqrt{85}}{85}}{1-\frac{36}{85}} = \frac{12\sqrt{85}}{85} \cdot \frac{85}{49}$$
$$= \frac{12\sqrt{85}}{49}$$

21. $$\cos\left(\sin^{-1}\frac{2}{3} + \tan^{-1}\frac{3}{2}\right)$$

Let $\alpha = \sin^{-1}\frac{2}{3}$ and $\beta = \tan^{-1}\frac{3}{2}$. Then $\sin\alpha = \frac{2}{3}$ and $\tan\beta = \frac{3}{2}$, and both α and β lie in quadrant I. Since $\sin\alpha = \frac{y_1}{r_1} = \frac{2}{3}$, let $y_1 = 2$ and $r_1 = 3$. Solve for x_1:
$$x_1^2 + 2^2 = 3^2$$
$$x_1^2 + 4 = 9$$
$$x_1^2 = 5$$
$$x_1 = \sqrt{5}$$

Thus, $\cos\alpha = \frac{x_1}{r_1} = \frac{\sqrt{5}}{3}$.

Since $\tan\beta = \frac{y_2}{x_2} = \frac{3}{2}$, let $x_2 = 2$ and $y_2 = 3$.

Solve for x_2:
$$2^2 + 3^2 = r_2^2$$
$$4 + 9 = r_2^2$$
$$r_2^2 = 13$$
$$r_2 = \sqrt{13}$$

Thus, $\sin\beta = \frac{y_2}{r_2} = \frac{3}{\sqrt{13}}$.

Therefore, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$$= \frac{\sqrt{5}}{3} \cdot \frac{2}{\sqrt{13}} - \frac{2}{3} \cdot \frac{3}{\sqrt{13}}$$
$$= \frac{2\sqrt{5}-6}{3\sqrt{13}}$$
$$= \frac{2\sqrt{13}\left(\sqrt{5}-3\right)}{39}$$

22. Let $\alpha = 75°$, $\beta = 15°$.

Since $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$,

$$\sin 75°\cos 15° = \frac{1}{2}\left[\sin(90°)+\sin(60°)\right]$$
$$= \frac{1}{2}\left[1+\frac{\sqrt{3}}{2}\right] = \frac{1}{4}\left(2+\sqrt{3}\right) = \frac{2+\sqrt{3}}{4}$$

23. $\sin 75° + \sin 15°$

$$= 2\sin\left(\frac{75°+15°}{2}\right)\cos\left(\frac{75°-15°}{2}\right)$$
$$= 2\sin(45°)\cos(30°) = 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{2}$$

24. $\cos 65°\cos 20° + \sin 65°\sin 20°$

$$= \cos(65°-20°) = \cos(45°) = \frac{\sqrt{2}}{2}$$

25. $4\sin^2\theta - 3 = 0$

$4\sin^2\theta = 3$

$\sin^2\theta = \frac{3}{4}$

$\sin\theta = \pm\frac{\sqrt{3}}{2}$

On the interval $[0, 2\pi)$, the sine function takes on a value of $\frac{\sqrt{3}}{2}$ when $\theta = \frac{\pi}{3}$ or $\theta = \frac{2\pi}{3}$. The sine takes on a value of $-\frac{\sqrt{3}}{2}$ when $\theta = \frac{4\pi}{3}$ and $\theta = \frac{5\pi}{3}$. The solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

26. $-3\cos\left(\frac{\pi}{2}-\theta\right) = \tan\theta$

$$-3\sin\theta = \tan\theta$$
$$0 = \frac{\sin\theta}{\cos\theta} + 3\sin\theta$$
$$0 = \sin\theta\left(\frac{1}{\cos\theta}+3\right)$$

$\sin\theta = 0$ or $\frac{1}{\cos\theta} + 3 = 0$

$\cos\theta = -\frac{1}{3}$

On the interval $[0, 2\pi)$, the sine function takes on a value of 0 when $\theta = 0$ or $\theta = \pi$. The cosine function takes on a value of $-\frac{1}{3}$ in the second and third quadrants when $\theta = \pi - \cos^{-1}\frac{1}{3}$ and $\theta = \pi + \cos^{-1}\frac{1}{3}$. That is $\theta \approx 1.911$ and $\theta \approx 4.373$. The solution set is $\{0, 1.911, \pi, 4.373\}$.

27. $\cos^2\theta + 2\sin\theta\cos\theta - \sin^2\theta = 0$

$$\left(\cos^2\theta - \sin^2\theta\right) + 2\sin\theta\cos\theta = 0$$
$$\cos(2\theta) + \sin(2\theta) = 0$$
$$\sin(2\theta) = -\cos(2\theta)$$
$$\tan(2\theta) = -1$$

The tangent function takes on the value -1 when its argument is $\frac{3\pi}{4} + k\pi$. Thus, we need

$$2\theta = \frac{3\pi}{4} + k\pi$$
$$\theta = \frac{3\pi}{8} + k\frac{\pi}{2}$$
$$\theta = \frac{\pi}{8}(3+4k)$$

On the interval $[0, 2\pi)$, the solution set is $\left\{\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$.

28. $\sin(\theta+1)=\cos\theta$

$\sin\theta\cos 1+\cos\theta\sin 1=\cos\theta$

$\dfrac{\sin\theta\cos 1+\cos\theta\sin 1}{\cos\theta}=\dfrac{\cos\theta}{\cos\theta}$

$\tan\theta\cos 1+\sin 1=1$

$\tan\theta\cos 1=1-\sin 1$

$\tan\theta=\dfrac{1-\sin 1}{\cos 1}$

Therefore, $\theta=\tan^{-1}\left(\dfrac{1-\sin 1}{\cos 1}\right)\approx 0.285$ or

$\theta=\pi+\tan^{-1}\left(\dfrac{1-\sin 1}{\cos 1}\right)\approx 3.427$

The solution set is $\{0.285, 3.427\}$.

29. $4\sin^2\theta+7\sin\theta=2$

$4\sin^2\theta+7\sin\theta-2=0$

Let $u=\sin\theta$. Then,

$4u^2+7u-2=0$

$(4u-1)(u+2)=0$

$4u-1=0$ or $u+2=0$

$4u=1$ $\quad u=-2$

$u=\dfrac{1}{4}$

Substituting back in terms of θ, we have

$\sin\theta=\dfrac{1}{4}$ or $\sin\theta=-2$

The second equation has no solution since $-1\le\sin\theta\le 1$ for all values of θ.

Therefore, we only need to find values of θ between 0 and 2π such that $\sin\theta=\dfrac{1}{4}$. These will occur in the first and second quadrants.

Thus, $\theta=\sin^{-1}\dfrac{1}{4}\approx 0.253$ and

$\theta=\pi-\sin^{-1}\dfrac{1}{4}\approx 2.889$.

The solution set is $\{0.253, 2.889\}$.

Chapter 7 Cumulative Review

1. $3x^2+x-1=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{1^2-4(3)(-1)}}{2(3)}=\frac{-1\pm\sqrt{1+12}}{6}=\frac{-1\pm\sqrt{13}}{6}$$

The solution set is $\left\{\dfrac{-1-\sqrt{13}}{6},\dfrac{-1+\sqrt{13}}{6}\right\}$.

2. Line containing points $(-2,5)$ and $(4,-1)$:

$m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{-1-5}{4-(-2)}=\dfrac{-6}{6}=-1$

Using $y-y_1=m(x-x_1)$ with point $(4,-1)$,

$y-(-1)=-1(x-4)$

$y+1=-1(x-4)$

$y+1=-x+4$

$y=-x+3$ or $x+y=3$

Distance between points $(-2,5)$ and $(4,-1)$:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(4-(-2))^2+(-1-5)^2}=\sqrt{6^2+(-6)^2}=\sqrt{72}=\sqrt{36\cdot 2}=6\sqrt{2}$$

Midpoint of segment with endpoints $(-2,5)$ and $(4,-1)$:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{-2+4}{2},\frac{5+(-1)}{2}\right)=(1,2)$$

3. $3x+y^2=9$

x-intercept: $3x+0^2=9$; $(3,0)$

$3x=9$

$x=3$

y-intercepts: $3(0)+y^2=9$; $(0,-3)$, $(0,3)$

$y^2=9$

$y=\pm 3$

Tests for symmetry:

x-axis: Replace y with $-y$: $3x+(-y)^2=9$

$$3x+y^2=9$$

Since we obtain the original equation, the graph is symmetric with respect to the x-axis.

y-axis: Replace x with $-x$: $3(-x)+y^2=9$

$$-3x+y^2=9$$

Since we do not obtain the original equation, the graph is not symmetric with respect to the y-axis.

Origin: Replace x with $-x$ and y with $-y$:

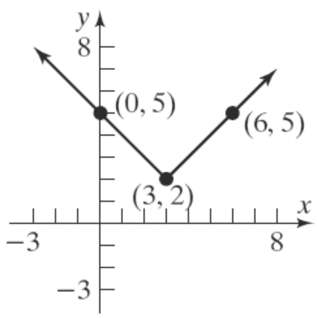
$$3(-x)+(-y)^2=9$$

$$-3x+y^2=9$$

Since we do not obtain the original equation, the graph is not symmetric with respect to the origin.

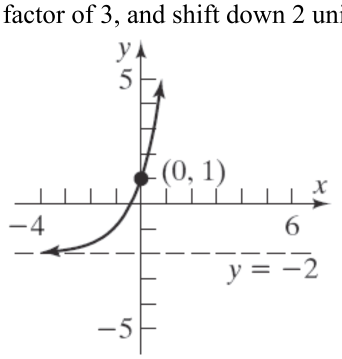
4. $y=|x-3|+2$

Using the graph of $y=|x|$, shift horizontally to the right 3 units and vertically up 2 units.



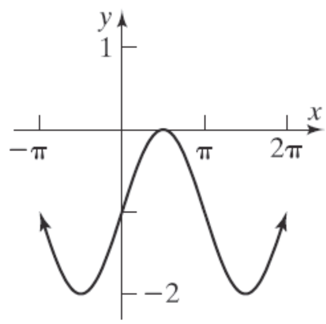
5. $y=3e^x-2$

Using the graph of $y=e^x$, stretch vertically by a factor of 3, and shift down 2 units.

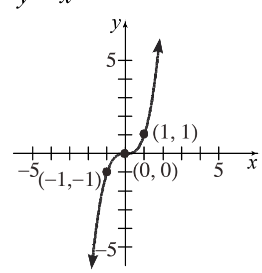


6. $y=\cos\left(x-\dfrac{\pi}{2}\right)-1$

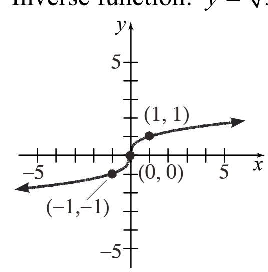
Using the graph of $y=\cos x$, horizontally shift to the right $\dfrac{\pi}{2}$ units, and vertically shift down 1 unit.

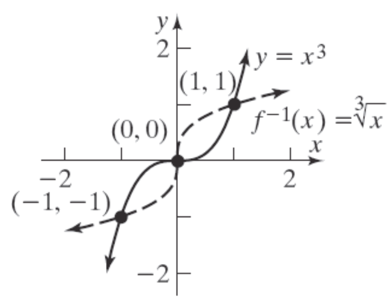


7. a. $y=x^3$

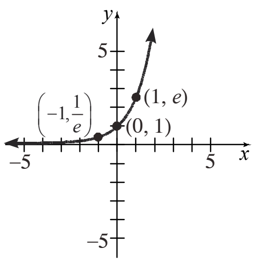


Inverse function: $y=\sqrt[3]{x}$

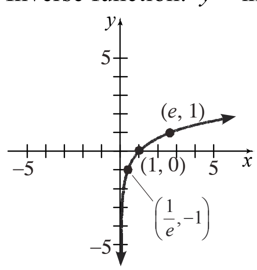


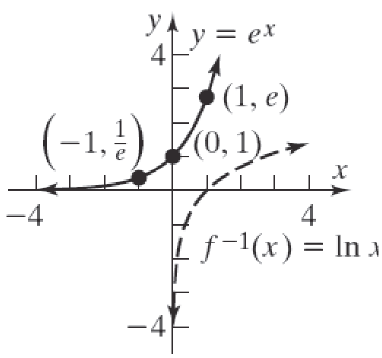


b. $y = e^x$

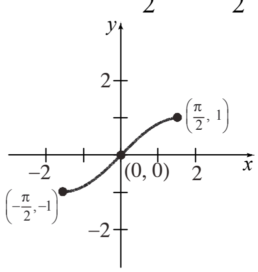


Inverse function: $y = \ln x$

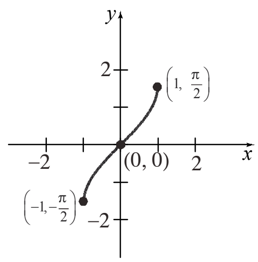


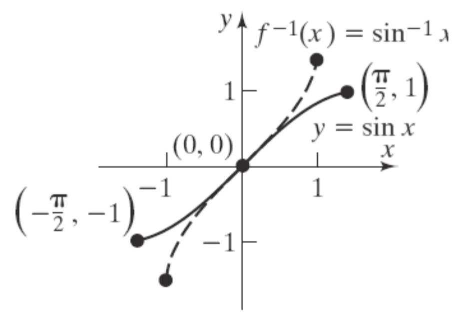


c. $y = \sin x,\ -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

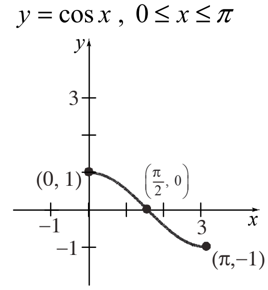


Inverse function: $y = \sin^{-1} x$

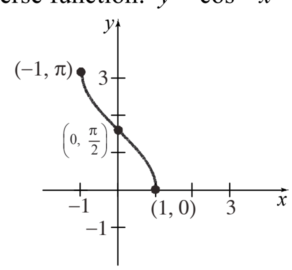


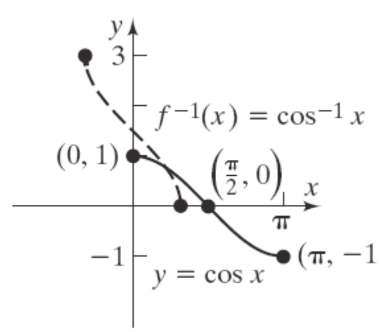


d. $y = \cos x,\ 0 \le x \le \pi$



Inverse function: $y = \cos^{-1} x$





8. $\sin\theta = -\frac{1}{3}, \pi < \theta < \frac{3\pi}{2}$, so θ lies in Quadrant III.

a. In Quadrant III, $\cos\theta < 0$

$$\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\left(-\frac{1}{3}\right)^2}$$
$$= -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}}$$
$$= -\frac{2\sqrt{2}}{3}$$

b. $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{3}}{-\frac{2\sqrt{2}}{3}}$$
$$= -\frac{1}{3}\left(-\frac{3}{2\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

c. $$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{1}{3}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$= \frac{4\sqrt{2}}{9}$$

d. $$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= \left(-\frac{2\sqrt{2}}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{8}{9} - \frac{1}{9} = \frac{7}{9}$$

e. Since $\pi < \theta < \frac{3\pi}{2}$, we have that $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$.

Thus, $\frac{1}{2}\theta$ lies in Quadrant II and $\sin\left(\frac{1}{2}\theta\right) > 0$.

$$\sin\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{2\sqrt{2}}{3}\right)}{2}}$$
$$= \sqrt{\frac{\frac{3+2\sqrt{2}}{3}}{2}} = \sqrt{\frac{3+2\sqrt{2}}{6}}$$

f. Since $\frac{1}{2}\theta$ lies in Quadrant II, $\cos\left(\frac{1}{2}\theta\right) < 0$.

$$\cos\left(\frac{1}{2}\theta\right) = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{2\sqrt{2}}{3}\right)}{2}}$$
$$= -\sqrt{\frac{\frac{3-2\sqrt{2}}{3}}{2}} = -\sqrt{\frac{3-2\sqrt{2}}{6}}$$

9. $\cos\left(\tan^{-1}2\right)$

Let $\theta = \tan^{-1}2$. Then $\tan\theta = \frac{y}{x} = \frac{2}{1}$,

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Let $x = 1$ and $y = 2$.

Solve for r: $r^2 = x^2 + y^2$
$$r^2 = 1^2 + 2^2$$
$$r^2 = 5$$
$$r = \sqrt{5}$$

θ is in quadrant I.

$$\cos\left(\tan^{-1}2\right) = \cos\theta = \frac{x}{r} = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

10. $\sin\alpha = \frac{1}{3}, \frac{\pi}{2} < \alpha < \pi$; $\cos\beta = -\frac{1}{3}, \pi < \beta < \frac{3\pi}{2}$

a. Since $\frac{\pi}{2} < \alpha < \pi$, we know that α lies in Quadrant II and $\cos\alpha < 0$.

$$\cos\alpha = -\sqrt{1-\sin^2\alpha}$$
$$= -\sqrt{1-\left(\frac{1}{3}\right)^2} = -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}}$$
$$= -\frac{2\sqrt{2}}{3}$$

b. $\pi < \beta < \frac{3\pi}{2}$, we know that β lies in Quadrant III and $\sin\beta < 0$.

$$\sin\beta = -\sqrt{1-\cos^2\beta}$$
$$= -\sqrt{1-\left(-\frac{1}{3}\right)^2}$$
$$= -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$$

c. $$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$
$$= \left(-\frac{2\sqrt{2}}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{8}{9} - \frac{1}{9} = \frac{7}{9}$$

d. $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= -\frac{2\sqrt{2}}{3}\left(-\frac{1}{3}\right) - \frac{1}{3}\left(-\frac{2\sqrt{2}}{3}\right)$$
$$= \frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = \frac{4\sqrt{2}}{9}$$

e. Since $\pi < \beta < \frac{3\pi}{2}$, we have that $\frac{\pi}{2} < \frac{\beta}{2} < \frac{3\pi}{4}$.

Thus, $\frac{\beta}{2}$ lies in Quadrant II and $\sin\frac{\beta}{2} > 0$.

$$\sin\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{1-\left(-\frac{1}{3}\right)}{2}}$$
$$= \sqrt{\frac{\frac{4}{3}}{2}} = \sqrt{\frac{4}{6}} = \frac{2}{\sqrt{6}} = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}$$

11. $f(x) = 2x^5 - x^4 - 4x^3 + 2x^2 + 2x - 1$

a. $f(x)$ has at most 5 real zeros.

Possible rational zeros:

$p = \pm 1;\ q = \pm 1, \pm 2;\ \frac{p}{q} = \pm 1, \pm\frac{1}{2}$

Using the Bounds on Zeros Theorem:

$f(x) = 2\left(x^5 - 0.5x^4 - 2x^3 + x^2 + x - 0.5\right)$

$a_4 = -0.5,\ a_3 = -2,\ a_2 = 1,\ a_1 = 1,\ a_0 = -0.5$

Max $\{1, |-0.5| + |1| + |1| + |-2| + |-0.5|\}$

$=$ Max $\{1, 5\} = 5$

$1 +$ Max $\{|-0.5|, |1|, |1|, |-2|, |-0.5|\}$

$= 1 + 2 = 3$

The smaller of the two numbers is 3. Thus, every zero of f must lie between -3 and 3.

Use synthetic division with –1:

$$\begin{array}{r|rrrrrr} -1 & 2 & -1 & -4 & 2 & 2 & -1 \\ & & -2 & 3 & 1 & -3 & 1 \\ \hline & 2 & -3 & -1 & 3 & -1 & 0 \end{array}$$

Since the remainder is 0, $x - (-1) = x + 1$ is a factor. The other factor is the quotient: $2x^4 - 3x^3 - x^2 + 3x - 1$.

Use synthetic division with 1 on the quotient:

$$\begin{array}{r|rrrrr} 1 & 2 & -3 & -1 & 3 & -1 \\ & & 2 & -1 & -2 & 1 \\ \hline & 2 & -1 & -2 & 1 & 0 \end{array}$$

Since the remainder is 0, $x - 1$ is a factor. The other factor is the quotient: $2x^3 - x^2 - 2x + 1$.

Factoring:

$$2x^3 - x^2 - 2x + 1 = x^2(2x-1) - 1(2x-1)$$
$$= (2x-1)(x^2-1)$$
$$= (2x-1)(x-1)(x+1)$$

Therefore,

$$f(x) = (2x-1)(x-1)^2(x+1)^2$$
$$= 2\left(x - \frac{1}{2}\right)(x-1)^2(x+1)^2$$

The real zeros are -1 and 1 (both with multiplicity 2) and $\frac{1}{2}$ (multiplicity 1).

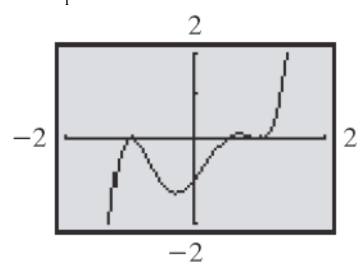
b. x-intercepts: 1, $\frac{1}{2}$, -1

y-intercept: -1

The intercepts are $(0,-1)$, $(1,0)$, $\left(\frac{1}{2},0\right)$, and $(-1,0)$

c. f resembles the graph of $y = 2x^5$ for large $|x|$.

d. Let $Y_1 = 2x^5 - x^4 - 4x^3 + 2x^2 + 2x - 1$

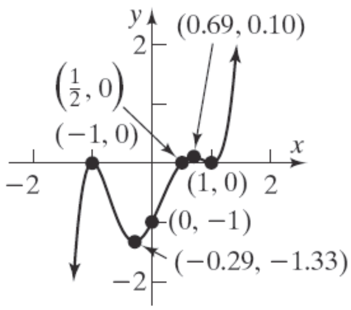


e. Four turning points exist. Use the MAXIMUM and MINIMUM features to locate local maxima at $(-1,0)$, $(0.69, 0.10)$ and local minima at $(1,0)$, $(-0.29,-1.33)$.

f. To graph by hand, we determine some additional information about the intervals between the x-intercepts:

Interval	$(-\infty,-1)$	$(-1,0.5)$	$(0.5,1)$	$(1,\infty)$
Test number	-2	0	0.7	2
Value of f	-45	-1	≈ 0.1	27
Location	Below x-axis	Below x-axis	Above x-axis	Above x-axis
Point	$(-2,-45)$	$(0,-1)$	$(0.7,0.1)$	$(2,27)$

f is above the x-axis for $(0.5,1)$ and $(1,\infty)$, and below the x-axis for $(-\infty,-1)$ and $(-1,0.5)$.



g. f is increasing on $(-\infty,-1]$, $[-0.29,0.69]$, and $[1,\infty)$. f is decreasing on $[-1,-0.29]$ and $[0.69,1]$.

12. $f(x)=2x^2+3x+1$; $g(x)=x^2+3x+2$

a.
$$f(x)=0$$
$$2x^2+3x+1=0$$
$$(2x+1)(x+1)=0$$
$$x=-\frac{1}{2} \text{ or } x=-1$$
The solution set is $\left\{-1,-\frac{1}{2}\right\}$.

b.
$$f(x)=g(x)$$
$$2x^2+3x+1=x^2+3x+2$$
$$x^2-1=0$$
$$(x+1)(x-1)=0$$
$$x=-1 \text{ or } x=1$$
The solution set is $\{-1, 1\}$.

c.
$$f(x)>0$$
$$2x^2+3x+1>0$$
$$(2x+1)(x+1)>0$$
$$f(x)=(2x+1)(x+1)$$
The zeros of f are $x=-\frac{1}{2}$ and $x=-1$

Interval	$(-\infty,-1)$	$\left(-1,-\frac{1}{2}\right)$	$\left(-\frac{1}{2},\infty\right)$
Test number	–2	–0.75	0
Value of f	3	–0.125	1
Conclusion	Positive	Negative	Positive

The solution set is $(-\infty,-1) \cup \left(-\frac{1}{2},\infty\right)$.

d.
$$f(x)\ge g(x)$$
$$2x^2+3x+1\ge x^2+3x+2$$
$$x^2-1\ge 0$$
$$(x+1)(x-1)\ge 0$$
$$p(x)=(x-1)(x+1)$$
The zeros of p are $x=-1$ and $x=1$.

Interval	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
Test number	–2	0	2
Value of p	3	–1	3
Conclusion	Positive	Negative	Positive

The solution set is $(-\infty,-1] \cup [1,\infty)$.

Chapter 7 Projects

Project I – Internet-based Project

Project II

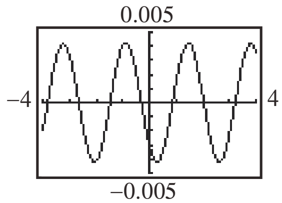
a. Amplitude = 0.00421 m

b. $\omega = 2.68$ radians/sec

c. $f=\frac{\omega}{2\pi}=\frac{2.68}{2\pi}\approx 0.4265$ vibrations/sec

d. $\lambda=\frac{2\pi}{k}=\frac{2\pi}{68.3}\approx 0.09199$ m

e. If $x = 1$, the resulting equation is $y = 0.00421\sin(68.3 - 2.68t)$. To graph, let $Y_1 = 0.00421\sin(68.3 - 2.68x)$.



f. Note: $(kx - \omega t) + (kx - \omega t + \phi) = 2kx - 2\omega t + \phi$ and $(kx - \omega t) - (kx - \omega t + \phi) = -\phi$.

$$y_1 + y_2 = y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi)$$
$$= y_m[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)]$$
$$= y_m\left[2\sin\left(\frac{2kx - 2wt + \phi}{2}\right)\cos\left(\frac{-\phi}{2}\right)\right]$$
$$= 2y_m \sin\left(\frac{2kx - 2wt + \phi}{2}\right)\cos\left(\frac{\phi}{2}\right)$$

g. $y_m = 0.0045$, $\phi = 2.5$, $\lambda = 0.09$, $f = 2.3$

Let $x = 1$:

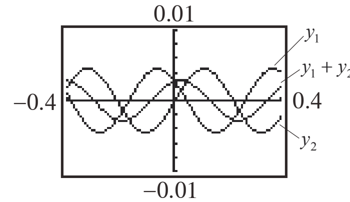
$$\lambda = 0.09 = \frac{2\pi}{k} \qquad f = 2.3 = \frac{\omega}{2\pi}$$
$$k = \frac{200\pi}{9} \approx 69.8 \qquad \omega = 4.6\pi \approx 14.45$$

$$y_1 = y_m \sin(kx - \omega t)$$
$$= 0.0045\sin(69.8 \cdot 1 - 14.45t)$$
$$= 0.0045\sin(69.8 - 14.45t)$$

$$y_2 = y_m \sin(kx - \omega t + \phi)$$
$$= 0.0045\sin(69.8 \cdot 1 - 14.45t + 2.5)$$
$$= 0.0045\sin(72.3 - 14.45t)$$

$$y_1 + y_2 = 2y_m \sin\left(\frac{2kx - 2\omega t + \phi}{2}\right)\cos\left(\frac{\phi}{2}\right)$$
$$= 2 \cdot 0.0045\sin\left(\frac{2 \cdot 69.8 \cdot 1 - 2 \cdot 14.45t + 2.5}{2}\right)\cos\left(\frac{2.5}{2}\right)$$
$$= 0.009\sin\left(\frac{142.1 - 28.9t}{2}\right)\cos(1.25)$$
$$= 0.009\sin(71.05 - 14.45t)\cos(1.25)$$

h. Let $Y_1 = 0.0045\sin(69.8 - 14.45x)$, $Y_2 = 0.0045\sin(72.3 - 14.45x)$, and $Y_3 = 0.009\sin(71.05 - 14.45x)\cos(1.25)$.



i. $y_m = 0.0045$, $\phi = 0.4$, $\lambda = 0.09$, $f = 2.3$

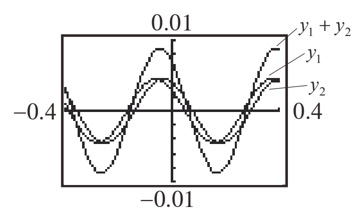
Let $x = 1$:

$$\lambda = 0.09 = \frac{2\pi}{k} \qquad f = 2.3 = \frac{\omega}{2\pi}$$
$$k = \frac{200\pi}{9} = 69.8 \qquad \omega = 4.6\pi = 14.45$$

$$y_1 = 0.0045\sin(69.8 - 14.45t)$$
$$y_2 = y_m \sin(kx - \omega t + \phi)$$
$$= 0.0045\sin(69.8 \cdot 1 - 14.45t + 0.4)$$
$$= 0.0045\sin(70.2 - 14.45t)$$

$$y_1 + y_2 = 2y_m \sin\left(\frac{2kx - 2\omega t + \phi}{2}\right)\cos\left(\frac{\phi}{2}\right)$$
$$= 2 \cdot 0.0045\sin\left(\frac{2 \cdot 69.8 \cdot 1 - 2 \cdot 14.45t + 0.4}{2}\right)\cos\left(\frac{0.4}{2}\right)$$
$$= 0.009\sin\left(\frac{140 - 28.9t}{2}\right)\cos(0.2)$$
$$= 0.009\sin(70 - 14.45t)\cos(0.2)$$

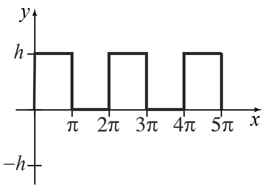
Let $Y_1 = 0.0045\sin(69.8 - 14.45x)$, $Y_2 = 0.0045\sin(70.2 - 14.45x)$, and $Y_3 = 0.009\sin(70 - 14.45x)\cos(0.2)$.



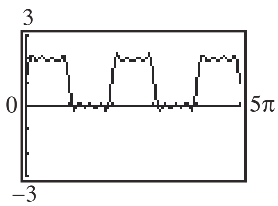
j. The phase shift causes the amplitude of $y_1 + y_2$ to increase from $0.009\cos(1.25) \approx 0.003$ to $0.009\cos(0.2) \approx 0.009$.

Project III

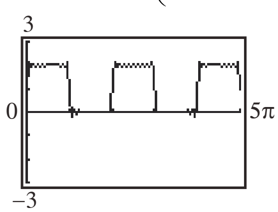
a.



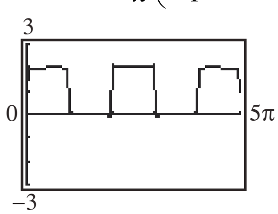
b. Let $Y_1 = 1 + \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7}\right)$



c. Let $Y_1 = 1 + \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin(3x)}{3} + \ldots + \frac{\sin(17x)}{17}\right)$



d. Let $Y_1 = 1 + \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin(3x)}{3} + \ldots + \frac{\sin(37x)}{37}\right)$



e. The best one is the one with the most terms.

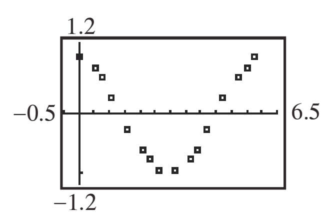
Project IV

a. $f(x) = \sin x$ (see table column 2)

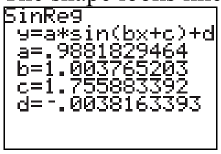
x	$f(x)$	$g(x)$	$h(x)$	$k(x)$	$m(x)$
0	0	0.954	−0.311	−0.749	6.085
$\frac{\pi}{6}$	$\frac{1}{2}$	0.791	−0.703	2.437	4.011
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	0.607	−1.341	1.387	−3.052
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	0.256	−0.978	0.588	−1.243
$\frac{\pi}{2}$	1	−0.256	−0.670	−0.063	0.413
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	−0.607	−0.703	0.153	8.507
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	−0.791	−0.623	2.380	−6.822
$\frac{5\pi}{6}$	$\frac{1}{2}$	−0.954	0	0.594	−2.695
π	0	−0.954	0.311	−0.817	1.536
$\frac{7\pi}{6}$	$-\frac{1}{2}$	−0.791	−0.117	−0.013	−5.248
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	−0.607	1.341	−1.387	3.052
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	−0.256	0.978	−0.588	1.243
$\frac{3\pi}{2}$	−1	0.256	0.670	0.063	−0.705
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	0.607	0.703	−0.306	
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	0.791	0.623		
$\frac{11\pi}{6}$	$-\frac{1}{2}$	0.954			
2π	1				

b. $g(x) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$ (see table column 3)

c.



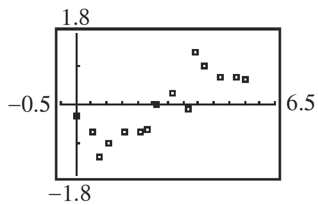
The shape looks like a sinusoidal graph.



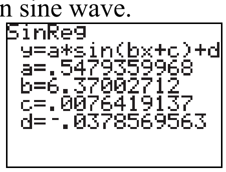
Rounding a, b, c, and d to the nearest tenth, we have that $y = \sin(x+1.8)$.

Barring error due to rounding and approximation, this looks like $y = \cos x$

d. $h(x) = \dfrac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i}$ (see table column 4)

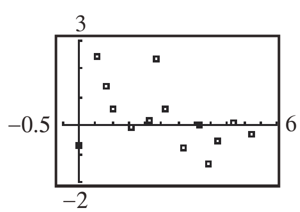


The shape is sinusoidal. It looks like an upside-down sine wave.

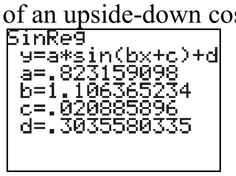


Rounding a, b, c, and d to the nearest tenth, we have that $y = 0.5\sin(6.4x)$.

e. $k(x) = \dfrac{h(x_{i+1}) - h(x_i)}{x_{i+1} - x_i}$ (see table column 5)



This curve is losing its sinusoidal features, although it still looks like one. It takes on the features of an upside-down cosine curve



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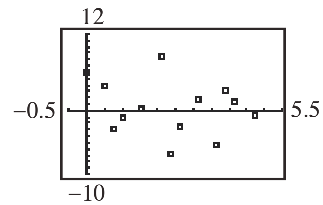
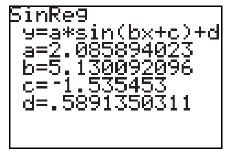
Rounding a, b, c, and d to the nearest tenth, we have that $y = 0.8\sin(1.1x) + 0.3$.

Note: The rounding error is getting greater and greater.

f. $m(x) = \dfrac{k(x_{i+1}) - k(x_i)}{x_{i+1} - x_i}$ (see table column 6)



The sinusoidal features are gone.



Rounding a, b, c, and d to the nearest tenth, we have that $y = 2.1\sin(5.1x - 1.5) + 0.6$.

g. It would seem that the curves would be less “involved”, but the rounding error has become incredibly great that the points are nowhere near accurate at this point in calculating the differences.