

## 2.1 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

## The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means “touching.” Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once, as in Figure 1(a). For more complicated curves this definition is inadequate. Figure 1(b) shows two lines  $l$  and  $t$  passing through a point  $P$  on a curve  $C$ . The line  $l$  intersects  $C$  only once, but it certainly does not look like what we think of as a tangent. The line  $t$ , on the other hand, looks like a tangent but it intersects  $C$  twice.

To be specific, let's look at the problem of trying to find a tangent line  $t$  to the parabola  $y = x^2$  in the following example.

**EXAMPLE 1** Find an equation of the tangent line to the parabola  $y = x^2$  at the point  $P(1, 1)$ .

**SOLUTION** We will be able to find an equation of the tangent line  $t$  as soon as we know its slope  $m$ . The difficulty is that we know only one point,  $P$ , on  $t$ , whereas we need two points to compute the slope. But observe that we can compute an approximation to  $m$  by choosing a nearby point  $Q(x, x^2)$  on the parabola (as in Figure 2) and computing the slope  $m_{PQ}$  of the secant line  $PQ$ . [A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

We choose  $x \neq 1$  so that  $Q \neq P$ . Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point  $Q(1.5, 2.25)$  we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

The tables in the margin show the values of  $m_{PQ}$  for several values of  $x$  close to 1. The closer  $Q$  is to  $P$ , the closer  $x$  is to 1 and, it appears from the tables, the closer  $m_{PQ}$  is to 2. This suggests that the slope of the tangent line  $t$  should be  $m = 2$ .

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \rightarrow P} m_{PQ} = m \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line (see Appendix B) to write the equation of the tangent line through  $(1, 1)$  as

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$$

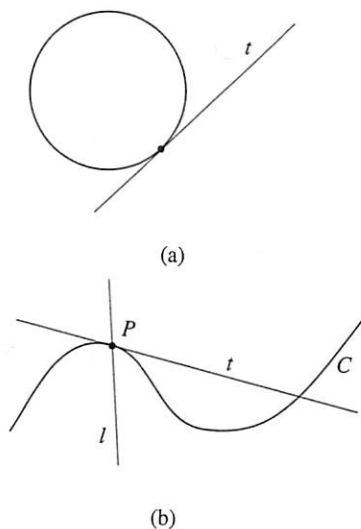


FIGURE 1

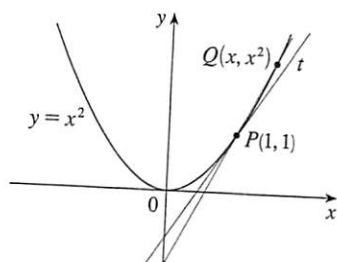


FIGURE 2

$x$	$m_{PQ}$
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

$x$	$m_{PQ}$
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

Figure 3 illustrates the limiting process that occurs in this example. As  $Q$  approaches  $P$  along the parabola, the corresponding secant lines rotate about  $P$  and approach the tangent line  $t$ .

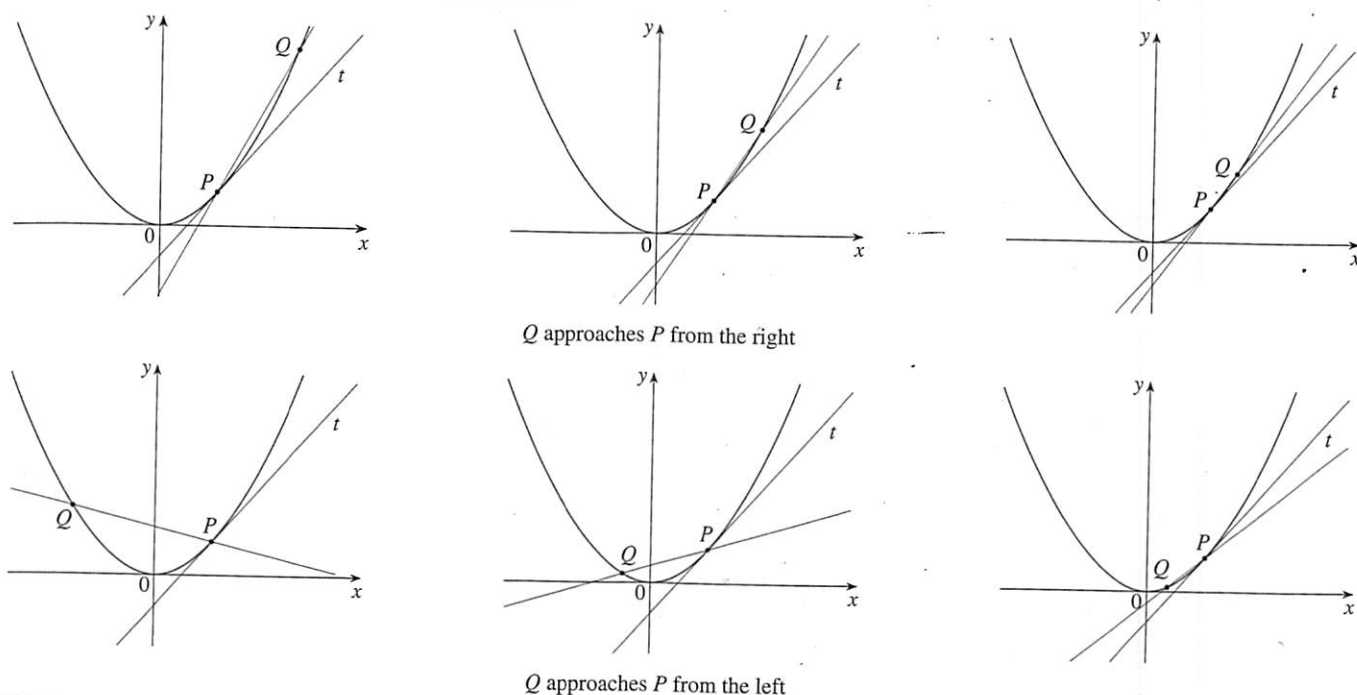


FIGURE 3

Many functions that occur in science are not described by explicit equations; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

$t$	$Q$
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

**V EXAMPLE 2 Estimating the slope of a tangent line from experimental data** The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the table describe the charge  $Q$  remaining on the capacitor (measured in microcoulombs) at time  $t$  (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where  $t = 0.04$ . [Note: The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

**SOLUTION** In Figure 4 we plot the given data and use them to sketch a curve that approximates the graph of the function.

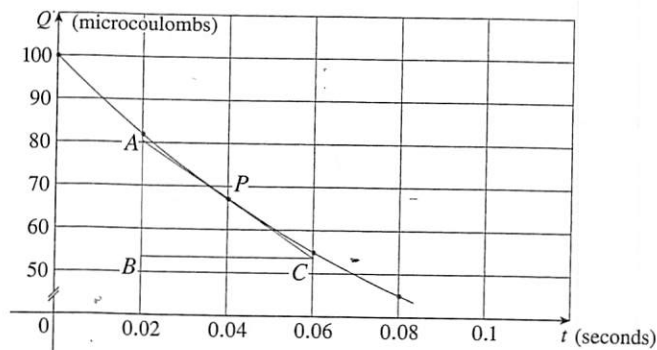


FIGURE 4

Given the points  $P(0.04, 67.03)$  and  $R(0.00, 100.00)$  on the graph, we find that the slope of the secant line  $PR$  is

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$

$R$	$m_{PR}$
(0.00, 100.00)	-824.25
(0.02, 81.87)	-742.00
(0.06, 54.88)	-607.50
(0.08, 44.93)	-552.50
(0.10, 36.76)	-504.50

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at  $t = 0.04$  to lie somewhere between  $-742$  and  $-607.5$ . In fact, the average of the slopes of the two closest secant lines is

$$\frac{1}{2}(-742 - 607.5) = -674.75$$

So, by this method, we estimate the slope of the tangent line to be  $-675$ .

Another method is to draw an approximation to the tangent line at  $P$  and measure the sides of the triangle  $ABC$ , as in Figure 4. This gives an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02} = -670$$

The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the flash bulb after 0.04 second is about  $-670$  microamperes.

### The Velocity Problem

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn't stay still for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.

**EXAMPLE 3 Velocity of a falling ball** Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

**SOLUTION** Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after  $t$  seconds is denoted by  $s(t)$  and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ( $t = 5$ ), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from  $t = 5$  to  $t = 5.1$ :

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s} \end{aligned}$$



Brand X Pictures/Jupiter Images/Fotosearch

The CN Tower in Toronto was the tallest freestanding building in the world for 32 years.

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leq t \leq 6$	53.9
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when  $t = 5$  is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at  $t = 5$ . Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 5) and we consider the points  $P(a, 4.9a^2)$  and  $Q(a + h, 4.9(a + h)^2)$  on the graph, then the slope of the secant line  $PQ$  is

$$m_{PQ} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

which is the same as the average velocity over the time interval  $[a, a + h]$ . Therefore the velocity at time  $t = a$  (the limit of these average velocities as  $h$  approaches 0) must be equal to the slope of the tangent line at  $P$  (the limit of the slopes of the secant lines).

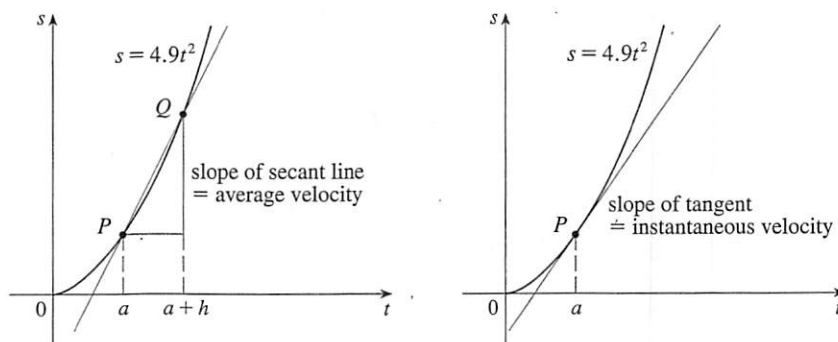


FIGURE 5

Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next four sections, we will return to the problems of finding tangents and velocities in Section 2.6.

## 2.1 Exercises

1. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume  $V$  of water remaining in the tank (in gallons) after  $t$  minutes.

$t$ (min)	5	10	15	20	25	30
$V$ (gal)	694	444	250	111	28	0

- (a) If  $P$  is the point  $(15, 250)$  on the graph of  $V$ , find the slopes of the secant lines  $PQ$  when  $Q$  is the point on the graph with  $t = 5, 10, 20, 25,$  and  $30$ .
- (b) Estimate the slope of the tangent line at  $P$  by averaging the slopes of two secant lines.
- (c) Use a graph of the function to estimate the slope of the tangent line at  $P$ . (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)
2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after  $t$  minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

$t$ (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of  $t$ .

- (a)  $t = 36$  and  $t = 42$       (b)  $t = 38$  and  $t = 42$   
 (c)  $t = 40$  and  $t = 42$       (d)  $t = 42$  and  $t = 44$

What are your conclusions?

3. The point  $P(1, \frac{1}{2})$  lies on the curve  $y = x/(1+x)$ .
- (a) If  $Q$  is the point  $(x, x/(1+x))$ , use your calculator to find the slope of the secant line  $PQ$  (correct to six decimal places) for the following values of  $x$ :
- (i) 0.5    (ii) 0.9    (iii) 0.99    (iv) 0.999  
 (v) 1.5    (vi) 1.1    (vii) 1.01    (viii) 1.001
- (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at  $P(1, \frac{1}{2})$ .
- (c) Using the slope from part (b), find an equation of the tangent line to the curve at  $P(1, \frac{1}{2})$ .
4. The point  $P(0.5, 0)$  lies on the curve  $y = \cos \pi x$ .
- (a) If  $Q$  is the point  $(x, \cos \pi x)$ , use your calculator to find the slope of the secant line  $PQ$  (correct to six decimal places) for the following values of  $x$ :
- (i) 0    (ii) 0.4    (iii) 0.49    (iv) 0.499  
 (v) 1    (vi) 0.6    (vii) 0.51    (viii) 0.501
- (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at  $P(0.5, 0)$ .

- (c) Using the slope from part (b), find an equation of the tangent line to the curve at  $P(0.5, 0)$ .
- (d) Sketch the curve, two of the secant lines, and the tangent line.

5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet  $t$  seconds later is given by  $y = 40t - 16t^2$ .
- (a) Find the average velocity for the time period beginning when  $t = 2$  and lasting
- (i) 0.5 second    (ii) 0.1 second  
 (iii) 0.05 second    (iv) 0.01 second
- (b) Estimate the instantaneous velocity when  $t = 2$ .
6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters  $t$  seconds later is given by  $y = 10t - 1.86t^2$ .
- (a) Find the average velocity over the given time intervals:
- (i)  $[1, 2]$     (ii)  $[1, 1.5]$     (iii)  $[1, 1.1]$   
 (iv)  $[1, 1.01]$     (v)  $[1, 1.001]$
- (b) Estimate the instantaneous velocity when  $t = 1$ .
7. The table shows the position of a cyclist.

$t$ (seconds)	0	1	2	3	4	5
$s$ (meters)	0	1.4	5.1	10.7	17.7	25.8

- (a) Find the average velocity for each time period:
- (i)  $[1, 3]$     (ii)  $[2, 3]$     (iii)  $[3, 5]$     (iv)  $[3, 4]$
- (b) Use the graph of  $s$  as a function of  $t$  to estimate the instantaneous velocity when  $t = 3$ .
8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion  $s = 2 \sin \pi t + 3 \cos \pi t$ , where  $t$  is measured in seconds.
- (a) Find the average velocity during each time period:
- (i)  $[1, 2]$     (ii)  $[1, 1.1]$   
 (iii)  $[1, 1.01]$     (iv)  $[1, 1.001]$
- (b) Estimate the instantaneous velocity of the particle when  $t = 1$ .
9. The point  $P(1, 0)$  lies on the curve  $y = \sin(10\pi/x)$ .
- (a) If  $Q$  is the point  $(x, \sin(10\pi/x))$ , find the slope of the secant line  $PQ$  (correct to four decimal places) for  $x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8,$  and  $0.9$ . Do the slopes appear to be approaching a limit?
- (b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at  $P$ .
- (c) By choosing appropriate secant lines, estimate the slope of the tangent line at  $P$ .